THE CARDINALITY OF FUZZY SETS AND CONTINUUM HYPOTHESIS (II)

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3. ON CH

For convenience, we review briefly for **CH**. It is well-known that every natural number is a cardinality; there does not exist another cardinality between n and n+1 which are two consecutive natural numbers. The least transfinite cardinality is that is unique infinite countable cardinality. If $\aleph_1$ is used for standing for the least noncountable cardinality, then there does not exist another cardinality between $\aleph_0$ and $\aleph_1$. So the chain of cardinalities should be

$$0,\ 1,\ 2, \cdots,\ \aleph_0,\ \aleph_1,\ \cdots\cdots$$

Here there does not exist another cardinality between any two consecutive natural numbers. On the other hand, $\aleph_0 < 2^{\aleph_0}$. So $\aleph_1 \leq 2^{\aleph_0}$. Is it true that $\aleph_1 = 2^{\aleph_0}$? Cantor guessed that the equality is true; namely there does not exist another cardinality between $\aleph_0$ and $2^{\aleph_0}$. This is well-known Continuum Hypothesis(**CH**).

CH has been generalized as the following: there does not exist another cardinality between α and 2^{α} for any transfinite cardinality α. This is Generalized Continuum Hypothesis(**GCH**).

By **GCH**, the chain of cardinality can be listed as:

$$0,\ 1,\ 2, \cdots,\ \aleph_0,\ 2^{\aleph_0},\ \ ,\ \alpha,\ 2^{\alpha},\ \cdots\cdots$$

In 1900, D.Hilbert promulgated 23 open problems, which the first was **CH**. In 1938, K.Gödel proved that if **ZFC** is compatible, then $\mathbf{ZFC} \nvdash \neg \mathbf{GCH}$; in 1963, P.Cohen proved that if **ZFC** is compatible, then $\mathbf{ZFC} \nvdash \mathbf{GCH}$. In general, **GCH** can not be decided in **ZFC**.

Just as a ruler and a pain of compasses can not divide any angle into three equal parts, **ZFC** can not decide that if **GCH** is true. We need new tools for dividing any angle into three equal parts; we also need new tools for solving **GCH**. Now for us analysing the well-known hypothesis from fuzzy sets, some new inspiration may be drawn from it.

Now we consider number systems. Natural numbers originated from the counting in production activities of mankind. Afterwords, as the development of productive forces, people had introduced negative numbers, integers and rational numbers. In rational number, four fundamental operations of arithmetic can be operated, as if to have no other numbers. But, because of the need of measuring the hypotenuse of a square with unit side length, 2 appeared. So people had to introduce irrational numbers, and formed real number system, which we have such appearance of mathematics. It is thus evident that a new notion may greatly promote the development of mathematics. This is also true for cardinality. Fuzzy sets have appeared since the need of describibing fuzzy phenomenon. Just as people mearsure the cardinality of Cantor' sets, we should also mearsure the cardinality of fuzzy sets. So we introduce F-cardinality as introduced irrational numbers. This will make cardinality system more perfect.

Are there no any other cardinalities between any two consecutive cardinalities? Of course, there were no any other numbers between any two consective intergers before rational numbers appeared; there were only rational numbers between any two rational numbers before irrational numbers appeared. Similarly, there are indeed no cardinalities between any two consecutive cardinalities before F-cardinality appears. Now this not true. For example, there exist F-cardinalities between 0 and 1.

Example 3.1 Let $X=\{x\}$ be a set with a single element and $A\in\mathcal{F}(X)$ with $A(x)=0.5$. Then $0=|\phi|<|A|<|X|=1$. In general, $\forall t\in(0,\ 1)$, if we put $A^t(x)=t$, then $0<|A^t|<1$. Noting $s\neq t\Longleftrightarrow|A^s|=|A^t|$, we have inserted $2^{\aleph_0}$ F-cardinalities. By using of the symbol $<n,t>$ introduced above section, we have $0 < <1,t> < 1$, $t\in(0,\ 1)$. Especially, $<1,0>=0$, $<1,1>=1$.

Now we try to insert F-cardinalities between any two consecutive natural numbers n and n+1.

Let $X=\{x_1,\ x_2,\ \cdots\ ,\ x_n\}(n\geq2)$ be a finite universe, p and q be natural numbers satisfying $p+q=n$. Taking a subset $X_1\subseteq X$ with $|X_1|=p$. Let $X_2=X-X_1$, then $|X_2|=q$. For any fixed $t\in[0,\ 1]$, define a fuzzy set $\overset{t}{A}\in\mathcal{F}(X)$:

$$\overset{t}{A}(x)=\begin{cases}1\ , & x\in X_1\\ t\ , & x\in X_2\end{cases}$$

called a finite double parts constant fuzzy set.

Denote $<p,\ q,\ t>=|\overset{t}{A}|$; clearly $<p,\ q,\ t>=<k,\ m,\ s>\Longleftrightarrow(p=k,\ q=m,\ s=t)$, which means that the symbol has unique meaning. Of course, the cardinalities of such kind of fuzzy sets are three-demensional quantity that can clearly express F-cardinality. Besides, above symbol $<n,\ t>$ is a special case of $<p,\ q,\ t>(p=0)$.

Example 3.2 Taking $<n,\ 1,\ t>$, when $t\in(0,\ 1)$, we have

$$n < <n,\ 1,\ t> < n+1$$

This means that we insert $2^{\aleph_0}$ F-cardinalities, and especially $<n,\ 1,\ 0>=n$, $<n,\ 1,\ 1>=n+1$.

Last we consider inserting F-cardinalities between two transfinite cardinalities.

Let α, β be two transfinite cardinalities with $\alpha<\beta$. Take a universe X such that $|X|=\beta$, and take subset $X_1 \subseteq X$ such that $|X_1|=\alpha$. Let $X_2=X-X_1$, then $|X_2|=\beta$. For any fixed $t\in[0, 1]$, define a fuzzy set $A^t \in \mathcal{F}(X)$:

$$A^t(x)=\begin{cases} 1 , & x\in X_1 \\ t , & x\in X_2 \end{cases}$$

called a transfinite double parts constent fuzzy set. Denote

$$\langle\alpha, \beta, t\rangle=|A^t|$$

Clearly, $\langle\alpha, \beta, t\rangle=\langle\gamma, \delta, s\rangle \Longleftrightarrow (\alpha=\gamma, \beta=\delta, t=s)$. So the symbol $\langle\alpha, \beta, t\rangle$ has unique meaning, which is also three-demensional quantity, and express clearly the F-cardinality of such kind of fuzzy sets by cardinalities and real numbers.

Remark: Since great difference between finite and infinite, $\langle p, q, t\rangle$ and $\langle\alpha, \beta, t\rangle$ are not completely unified. For example, $\alpha<\beta$, but that $p<q$ is not true. However, $p+q=n=|X|$, $\alpha+\beta=\beta=|X|$. So from this point, $\langle p, q, t\rangle$ is basicly a special case of $\langle\alpha, \beta, t\rangle$.

Example 3.3 Taking $\langle \text{æ}_0, 2^{\text{æ}_0}, t\rangle$, when $t\in(0, 1)$, we have

$$\text{æ}_0 < \langle \text{æ}_0, 2^{\text{æ}_0}, t\rangle < 2^{\text{æ}_0}$$

Namely, we insert $2^{\text{æ}_0}$ F-cardinalities between æ_0 and $2^{\text{æ}_0}$. Esprcially, $\langle \text{æ}_0, 2^{\text{æ}_0}, 0\rangle=\aleph_0$ and $\langle \text{æ}_0, 2^{\text{æ}_0}, 1\rangle=2^{\text{æ}_0}$.

Example 3.4 Let α be a transfinite cardinality. Taking $\langle\alpha, 2^{\alpha}, t\rangle$, when $t\in(0, 1)$. We can also insert $2^{\text{æ}_0}$ F-cardinalities:

$$\alpha < \langle\alpha, 2^{\alpha}, t\rangle < 2^{\alpha}$$

Especially, $<\alpha, 2^{\alpha}, 0> = \alpha$, $<\alpha, 2^{\alpha}, 1> = 2^{\alpha}$.

How many F-cardinalities are there between two consective cardinalities?

Theorem 3.1 There are just $2^{\aleph_0}$ F-cardinalities between two consecutive finite cardinalities, and 2^{β} F-cardinalities between two consective transfinite cardinalities($\alpha < \beta$).

4. THE SUM OF F-CARDINALITIES

Definition 4.1 Let $\{\alpha_t\}_{t\in T}$ be a class of F-cardinalities and $\{A^{(t)}\}_{t\in T}$ be a class of fuzzy sets which $A^{(t)}\neq A^{(s)}$ when $t\neq s$, such that $|A^{(t)}|=\alpha_t$, $t\in T$. If $A = \bigcup_{t\in T} A^{(t)}$, then $\alpha=|A|$ is called the sum of F-cardinalities α_t, $t\in T$, denoted by

$$\alpha = \sum_{t\in T} \alpha_t$$

Theorem 4.1 For any three F-cardinalities α, β, γ, we have

1) $\alpha+0=\alpha$

2) $\alpha+\beta=\beta+\alpha$

3) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$

4) $\alpha\leq\beta \Longrightarrow \alpha+\gamma\leq\beta+\gamma$

5) $\alpha\leq\alpha+\beta$, $\beta\leq\alpha+\beta$

Corollary 1. Let α, β, γ and δ be four F-cardinalities. If $\alpha\leq\gamma$, $\beta\leq\delta$, then $\alpha+\beta\leq\gamma+\delta$.

Remark. The result can be easily extended to infinite sum Σ: Let $\{\alpha_t\}_{t\in T}$, $\{\beta_t\}_{t\in T}$ be two classes of F-cardinalities. If $(\forall t\in T)(\alpha_t\leq\beta_t)$, then $\sum_{t\in T} \alpha_t \leq \sum_{t\in T} \beta_t$.

Let A be a fuzzy set. That $\mathrm{Ker}A=A_1=\{x|A(x)=1\}$ is called the Kernel of A.

Clearly, $|\mathrm{Ker}A| \le |A| \le |\mathrm{supp}A|$. $|\mathrm{Ker}A|$ and $|\mathrm{supp}A|$ are respectively called the sharp upper bound and the sharp lower bound.

Corollary 2. If $\{A^{(t)}\}_{t\in T}$ is a class of F-cardinalities which $A^{(t)} \cap A^{(s)} = \phi$ when $t \ne s$, then

$$\sum_{t\in T} |\mathrm{Ker}A^{(t)}| \le \sum_{t\in T} |A^{(t)}| \le \sum_{t\in T} |\mathrm{supp}A^{(t)}|$$

Definition 4.2 Let A be a fuzzy set. If there exists $\lambda \in [0, 1)$, such that $|A_\lambda|$ is a transfinite cardinality, then $|A|$ is called a transfinite F-cardinality, or else a finite F-cardinality.

Clearly. $|A|$ is a transfinite F-cardinality iff $|\mathrm{supp}A|$ is a transfinite cardinality.

Now we have a problem: Does transfinite F-cardinality satisfy absorption law for sum? Namely, if α is a transfinite F-cardinality, then α satisfies that $\alpha+\alpha=\alpha$.

The answer is negative. For example, A and B are defined as the following:

$$A(x)=\begin{cases} 1, & x=1 \\ 0.5, & x\in[0,\ 1) \end{cases} \qquad B(y)=\begin{cases} 1, & y=3 \\ 0.5, & y\in[2,\ 3) \end{cases}$$

Clearly $A \cap B = \phi$ and $|A|=|B|$. Let $|A|=\alpha$. It is easy to know that $|A \cup B| = \alpha+\alpha > \alpha$.

So we should intensify the definition.

Definition 4.3 For any fuzzy set A which $|A|$ is transfinite, if A satisfies the condition: $(\forall\lambda\in(0,\ 1))(A_\lambda \ne \phi \Longrightarrow |A_\lambda|$ is transfinite), then A is called a strong transfinite fuzzy set and $|A|$ called a strong transfinite F-cardinality.

Theorem 4.2 If α is a strong transfinite F-cardinality, then

$\alpha+\alpha=\alpha$.

Remark: By inductive method it is easy to prove that if α is a strong transfinite F-cardinality, then $\sum_{i=1}^{\infty} \alpha = \alpha$.

From this we have proved the absorption law with respect to sum. The more generalized absorption law is the following result.

Corollary. Let α be a strong transfinite F-cardinality and β be any fuzzy set. If $\alpha\leq\beta$, then $\alpha+\beta=\alpha$.

Theorem 4.3 Given fuzzy sets class $\{A^{(t)}\}_{t\in T}$, we have

$$\Big|\bigcup_{t\in T} A^{(t)}\Big| \leq \sum_{t\in T} |A^{(t)}|$$

5. THE PRODUCT OF F-CARDINALITIES

In preparation for the following, we discuss the cartesian product of fuzzy set in the first place.

Given a class of fuzzy sets $A^{(t)}\in\mathcal{F}(X_t)$, $t\in T$, denote

$$\prod_{t\in T} A^{(t)} := \bigcup_{\lambda\in[0,\ 1]} \lambda \quad \Big(\prod_{t\in T} A_{\lambda}^{(t)}\Big)$$

called cartesian product of fuzzy sets class $\{A^{(t)}\}_{t\in T}$, where

$$\prod_{t\in T} A_{\lambda}^{(t)} = \{f_{\lambda} \mid f_{\lambda}: T \longrightarrow \bigcup_{t\in T} A_{\lambda}^{(t)},\ f_{\lambda}(t)\in A_{\lambda}^{(t)},\ t\in T\}$$

It is easy to see that $\prod_{t\in T} A^{(t)}\in\mathcal{F}(\prod_{t\in T} X_t)$, where

$$\prod_{t\in T} X_t = \{f \mid f: T \longrightarrow \bigcup_{t\in T} X_t\ ,\ f(t)\in X_t\ ,\ t\in T\}$$

Clearly $\prod_{t\in T} A_{\lambda}^{(t)} \subseteq \prod_{t\in T} X_t$

Lemma 5.1 $\Big(\prod_{t\in T} A^{(t)}\Big)(f) = \bigwedge_{t\in T} A^{(t)}(f(t))$

Lemma 5.2. 1) $\Big(\prod_{t\in T} A^{(t)}\Big) = \prod A_{\lambda}^{(t)}$

2) $(\prod_{t\in T} A^{(t)})_\lambda \subseteq \prod_{t\in T} A_\lambda^{(t)}$

Remark. When T is finite set, 2) will become a equality.

Lemma 5.3 Let A, B, C, and D be fuzzy sets. Then

1) $A\times(B\cup C)=(A\times B)\cup(A\times C)$

2) $A\times(B\cap C)=(A\times B)\cap(A\times C)$

3) $A\times B=\phi \Longleftrightarrow A=\phi$ or $B=\phi$

4) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$

5) $A\subseteq C$ and $B\subseteq D \Longrightarrow A\times B\subseteq C\times D$

6) $A\times A=B\times B \Longrightarrow A=B$

Let $\{A^{(t)}\}_{t\in T}$ be a class of fuzzy sets satisfying that $A^{(t)}\cap A^{(s)}=\phi$ when $t\neq s$. If $\bigcup_{t\in T} A^{(t)}=A$, then $\{A^{(t)}\}_{t\in T}$ is called direct union decomposition of A.

Lemma 5.4 If $\{A^{(i)}\}_{i\in I}$ and $\{B^{(j)}\}_{j\in J}$ is respectively a direct union decomposition of A and B, then $\{A^{(i)}\times B^{(j)}\}_{(i,j)\in I\times J}$ is a direct union decomposition of $A\times B$.

Lemma 5.5 $|A|=|A'|$ and $|B|=|B'| \Longrightarrow |A\times B|=|A'\times B'|$

Definition 5.1 Let α, β, be F-cardinalities, and A, B be fuzzy sets with $\alpha=|A|$, $\beta=|B|$. $\gamma=|A\times B|$ is called the direct product of α and β, denoted by $\gamma=\alpha\beta$.

Remark. From Lemma 5.5, $\alpha\beta$ in the definition does not depend on the selection of fuzzy sets A and B. Besides, by the definition, we have $|A\times B|=|A||B|$.

Theorem 5.1 Let α, β and γ be any three F-cardinalities. We have

1) $\alpha 0=0$, $\alpha 1=\alpha$;

2) $(\alpha\beta)\gamma=\alpha(\beta\gamma)$;

3) $\alpha\beta=\beta\alpha$;

4) $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$.

Corollary. Let α, $\beta_t(t\in T)$ be F-cardinalities. We have the general distributive law:

$$\alpha \sum_{t\in T} \beta_t = \sum_{t\in T} (\alpha\beta_t)$$

Lemma 5.6 $\alpha\le\beta \Longrightarrow \alpha\gamma\le\beta\gamma$

Theorem 5.2 If α is a strong transfinite F-cardinality, then $\alpha\alpha=\alpha$.

Corollary. Let α be a strong transfinite F-cardinality and β be any F-cardinality. If β satisfy that $1\le\beta\le\alpha$, then $\alpha\beta=\alpha$.

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