DIAMETERS AND DISTANCES IN FUZZY SPACES.

by

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1. Introduction.

In this paper we examine some connections between pointless pseudometric space theory as defined in Weihrauch and Schreiber [1981] and Gerla [1990] and fuzzy set theory. In particular, we consider two pointless pseudometric spaces in wich the diameters are energy and entropy functions, respectively. Recall that a *pointless pseudometric space*, briefly *p-p-m-space*, is any structure $\mathcal{R}=(R,\leq,\delta,|\ |)$ where $(R,\leq)$ is a partial order and $|\ |:R\to[0,\infty]$, $\delta:R\times R\to[0,\infty)$ are functions such that, for every $x,y,z\in R$

A1 $x\geq y \Rightarrow |x|\geq|y|$; **A2** $x\geq y \Rightarrow \delta(y,z)\geq\delta(z,x)$

A3 $\delta(x,x)=0$; **A4** $\delta(x,y) \leq \delta(x,z) + \delta(z,y) + |z|$.

We call *regions* the elements of R, *inclusion* the relation $\leq$, *distance* between two regions x and y the number $\delta(x,y)$ and *diameter* of x the number $|x|$. Notice that A2 is equivalent to say that δ is symmetric and decreasing i.e.

$$\delta(x,y)=\delta(y,x) \quad ; \quad x\leq x'\ ,\ y\leq y' \Rightarrow \delta(x,y)\leq\delta(x',y') \quad .$$

From A1-A4 it follows that if a region z is included both in x and y then $\delta(x,y)=0$. In particular, if $x\leq y$ then $\delta(x,y)=0$ and the existence of a minimum in $(R,\leq)$ entails that δ is constantly equal to zero. So we assume that no minimum in R exists. If x and y are bounded, i.e. $|x|\neq\infty$ and $|y|\neq\infty$, we set $e(x,y) = \delta(x,y) + |x|/2 + |y|/2$. It is immediate that

$$e(x,y)=e(y,x) \ ; \ e(x,x)=|x| \ ; \ e(x,y)\leq e(x,z)+e(z,y) \ .$$

The theory of the p-p-m-spaces generalizes the theory of the pseudometric spaces; namely, the pseudometric spaces are the p-p-m-spaces for which every region is an atom with diameter equal to zero (in this case $\leq$ becomes the identity relation). We obtain a p-p-m-space by setting R equal to a class of nonempty subsets of a pseudometric space (M,d), $\leq$ the inclusion relation and δ and $| \, |$ the usual distance and diameter functions defined by

(1) $\delta(X,Y)=\text{Inf}\,\{d(x,y)/x\in X,\ y\in Y\}$; $|X| = \text{Sup}\,\{d(x,y)/x\in X,\ y\in X\}$.

We call *canonical* any space of this type. A canonical p-p-m-space is furnished by interval analysis where R is the set of closed intervals of the real line.

The points in a p-p-m-space $\mathcal{R}$ are defined as follows: we call a *Cauchy sequence* a sequence $(p_n)_{n\in N}$ of bounded regions such that

(2) $$\forall\varepsilon>0\ \exists\nu\in N\ \forall h\geq\nu\,\forall k\geq\nu\ e(p_h,p_k)<\varepsilon,$$

Decreasing sequences with vanishing diameters are examples of Cauchy sequences. Indeed in this case $\delta(p_h,p_k)=0$ for every h and k and (2) is satisfied. Assume that S is nonempty and define $d:S\times S\to[0,\infty)$ by setting, for every pair $p=(p_n)_{n\in N}$ and $q=(q_n)_{n\in N}$ of elements of S,

(3) $$d(p,q) = \lim e(p_n,q_n)\ ,$$

then it is easy to prove that (S,d) is a pseudometric space. We denote by $M(\mathcal{R})=(IP,d)$ the related *quotient* that is the metric space obtained as a quotient of (S,d) modulo the relation $\equiv$ defined by setting $p\equiv q$ if and only if $d(p,q)=0$. So a *point* is a class $[p]=\{q\in S/d(q,p)=0\}$, and $d:IP\times IP\to[0,\infty)$ is defined putting $d([p],[q]) = d(p,q)$.

A different definition of point can be obtained by confining ourselves to the class S_1 of decreasing sequences of bounded regions with vanishing diameters. We denote by $M_1(\mathcal{R})$ the metric space so obtained. $M_1(\mathcal{R})$ is different from $M(\mathcal{R})$, in general. If $\mathcal{R}$ is a metric space, then $M(\mathcal{R})$ is its completion, while $M_1(\mathcal{R})$ coincides with $\mathcal{R}$, obviously. If $\mathcal{R}$ is the canonical p-p-m-space of the open intervals of the rational number set, then $M_1(\mathcal{R})$ is equal to $M(\mathcal{R})$ and both are equal to the real line.

2. Fuzzy subsets in a metric space.

In this Section we will extend the classical diameter and distance functions to the fuzzy subsets of a pseudometric space (M,d) (see Gerla and Volpe [1986]). Let s and s' be two fuzzy subsets of M, i.e. two functions from M into the interval [0,1], and set

(4) $\delta^*(s,s')= \int_0^1 \delta(C(s,\alpha),C(s',\alpha))d\alpha$; $|s|^*= \int_0^1 |C(s,\alpha)|d\alpha$.

where $C(s,\alpha)$ is the cut $\{x\in X/s(x)\geq\alpha\}$ and δ and $|\ |$ are defined by (1) .

Notice that the diameter is an energy measure as defined in De Luca and Termini [1972] and that if $d=\mathrm{Sup}\{s(x)\wedge s'(x)/x\in M\}$ then

$$\delta^*(s,s')= \int_d^1 \delta(C(s,\alpha),C(s',\alpha))d\alpha .$$

As usual, we set $s\leq s'$ provided that $s(x)\leq s'(x)$ for every $x\in M$.

Proposition 1. Let R be a class of nonempty fuzzy subsets of M, then the structure $\mathcal{R}=(R,\leq,\delta^*,|\ |^*)$ defined by (4) is a p-p-m-space.

Proof. A1, A2 and A3 are immediate. Let s, s', t be nonempty fuzzy subsets, then by integrating both the sides of the inequality

$$\delta(C(s,\alpha),C(s',\alpha)) \leq \delta(C(s,\alpha),C(t,\alpha)) + \delta(C(t,\alpha),C(s',\alpha)) + |C(t,\alpha)|$$

we obtain A4. □

Proposition 1 enables us to consider two interesting cases of p-p-m-spaces .

Fuzzy numbers. Recall that a fuzzy number is any fuzzy subset $n:R\rightarrow[0,1]$ of the real line such that every closed cut is a nonempty closed interval. Since the real line is a metric space, the class of fuzzy numbers define a p-p-m-space extending the space of the interval numbers. It is interesting to observe that the diameter of a fuzzy number n is $\int_R n\, d\mu$. As an example, confine ourselves to the class of *triangular* fuzzy numbers $n_{c,r}$ where

$$n_{c,r}(x)=\begin{cases}(x-c+r)/r & \text{if} \quad c-r\le x\le c \ ;\\ (c-x+r)/r & \text{if} \quad c\le x\le c+r \ ;\\ 0 & \text{otherwise.}\end{cases}$$

Then there is no difficulty to prove that the diameter of $n_{c,r}$ is r and

$$\delta(n_{c,r},n_{c',r'})=\begin{cases}|c-c'| - (r+r')/2 & \text{if} \quad r+r'\le|c-c'| \ ;\\ (c-c')^2/(2(r+r')) & \text{otherwise.}\end{cases}$$

Flou subsets. Consider the particular case of the *flou subsets* , that is the case in which the possible true values are 0,1/2 and 1. By identifying each flou subset s with the pair (C(s,1),C(s,1/2)) of the related cuts, the p-p-m-space of Proposition 1 becomes as follows.

- R is a set of pair (E,F) with $E\subseteq F$ and E nonempty ;
- $(E,F)\le(E',F')$ if and only if $E\subseteq E'$ and $F\subseteq F'$;
- $\delta^*((E,F),(E',F'))=1/2\,[\delta(E,E')+\delta(F,F')]$;
- $|(E,F)|^*=1/2(|E|+|F|)$.

3. Entropy, diameters, distances.

A rather general method to build up examples of p-p-spaces in fuzzy set theory is the following (see Gerla [1991]c). Let S be a set, **A** an algebra of subsets of S, μ:**A**→R a measure. As it is usual, μ can be extended to the class $\mathfrak{M}$ of measurable fuzzy subsets of S by setting $\mu(s)=\int s\,d\mu$ for every s in $\mathfrak{M}$.

We call a *continuous p-p-m-space in [0,1]* any p-p-m-space $([0,1],\le',\delta',|\ |')$ such that δ' and $|\ |'$ are measurable maps. Then it is possible to extend $\le'$, δ' and $|\ |'$ to $\mathfrak{M}$ by putting, for every $s,t\in\mathfrak{M}$,

(5) $s\le t \iff s(x)\le' t(x) \ \ \forall x\in S$; $\delta(s,t)=\mu(\delta'(s,t))$; $|s|=\mu(|s|)$

where $\delta'(s,t)$ and $|s|'$ are the fuzzy subsets defined by $(\delta'(s,t))(x)=\delta'(s(x),t(x))$ and $|s|'(x)=|s(x)|'$.

Proposition 2. Let μ be a probability measure on S, $\mathfrak{M}$ the class of measurable fuzzy subsets of S and $([0,1],\leq',\delta',|\ |')$ a continuous p-p-m-space. Then the structure $(\mathfrak{M},\leq,\delta,|\ |)$ defined by (5) is a p-p-m-space.

It is not too difficult to find examples of continuous p-p-m-spaces. As an example, very interesting spaces are related with the notions of sharpness relation and of entropy (De Luca and Termini [1972]). Recall that the *sharpness relation* $\leq_S$ is an order relation in [0,1] defined by setting $\alpha\leq_S\beta$ if and only if

$$\beta < 1/2 \Rightarrow \alpha\leq\beta \quad \text{and} \quad \beta>1/2 \Rightarrow \alpha\geq\beta .$$

This relation is extended to the class F(S) of fuzzy subsets of S by setting $f \leq_S g$ provided that $f(x) \leq_S g(x)$ for every $x\in S$. The maximum in F(S) with respect to $\leq_S$ is the completely undefined fuzzy subset $i:X\rightarrow[0,1]$, i.e. the fuzzy set such that $i(x)=1/2$ for every $x\in S$. The atoms coincide with the classical subsets of S. An *entropy* is a map $h: F(S)\rightarrow[0,1]$ increasing with respect to $\leq_S$ such that $e(i)=1$ and $e(s)=0$ for every classical subset s. The entropies are measures of the degree of fuzzyness of the fuzzy subsets.

Now, a continuous p-p-m-spaces in obtained by setting $\leq'$ equal to the sharpness relation, $|\alpha|'=2\cdot(\alpha\wedge\sim\alpha)$ and

$$\delta'(\alpha,\beta) = \begin{cases} \alpha\vee\beta-\alpha\wedge\beta & \text{if} \quad \alpha\wedge\beta < 1/2 < \alpha\vee\beta \\ 0 & \text{otherwise.} \end{cases}$$

The related p-p-m-space is a space of fuzzy subsets such that

- the inclusion relation is the sharpness relation;
- the distance between two fuzzy subsets is the measure of the simmetric difference : $\delta(s,s')=\mu(s\nabla s')$;
- the diameter is the well known entropy : $|s| =2\mu(s\wedge\sim s')$
- the points coincide with the classical subsets.

This space generalizes a well know pseudometric spaces associated to a measure space (**A**,μ). Namely it is well know that, given a measure space (**A**,μ), the equality $d(X,Y)=\mu(X\nabla Y)$, where $X\nabla Y=(X-Y)\cup(Y-X)=X\cup Y-X\cap Y$ is the symmetric difference, defines in **A** a structure of pseudometric space (**A**,d).

4. First order theories and pointless spaces.

In the case of the flou subsets, the p-p-m-spaces of Proposition 3 are defined as follows.

- R is the class of flou subsets (E,F) with $E\in \mathbf{A}-\{\emptyset\}$ and $F\in \mathbf{A}$;
- $(E,F)\leq(E',F')$ if and only if $E'\subseteq E$ and $F\subseteq F'$;
- $\delta((E,F),(E',F'))=\mu(E\cup E'-F\cap F')$;
- $|(E,F)|=\mu(F-E)$.

An interesting class of flou subsets of the set F of sentences of a first order language $\mathcal{L}$ is obtained by the theories of $\mathcal{L}$. Recall that a *theory* is any consistent set T of sentences closed with respect to logical deductions. Indeed it is very natural to associate to every theory T the flou subset (T,c(T)) where c(T) is the set of sentences consistent with T. In other words, we assign the value 1 to the provable formulas, the value 0 to the disprovable formulas and 1/2 to the undecidable formulas.

In order to semplify the matematical treatment, we prefer to substitute F with the Lindenbaum algebra **B** and therefore to identify a theory T with a filter and consequently c(T) with $\{x\in B/ \sim x\notin T\}$. Moreover the measures in **B** can be introduced as follows.

We call a *weight* any function $r:\mathbf{B}\to[0,1]$ such that

(6) $$\Sigma_{\alpha\in B}\, r(\alpha)=1 \qquad \text{and} \qquad r(0)=0 \;;$$

and we interpret $r(\alpha)$ as the *degree of importance* of the sentence α. As usual, a measure $\mu:\mathcal{P}(B)\to[0,1]$ is obtained by setting, for every subset X of **B**, $\mu(X)=\sum_{\alpha\in X} r(\alpha)$. We call *relevant* every sentence α such that $r(\alpha)\neq 0$. We say that two theories T and T' *fight over the sentence* α , provided that either $\alpha\in T$ and $\neg\alpha\in T'$ or $\neg\alpha\in T$ and $\alpha\in T'$. We say that α

decidable in T, if either $\alpha \in T$ or $\neg\alpha \in T$.

Coming back to the p-p-m-spaces, it is immediate to prove the following proposition.

Proposition 3. Let $\mathcal{T}$ be a class of theories and set

(7) $$\delta(T,T') = \sum\{r(\alpha) / \text{ T and T' fight over } \alpha\} \quad ;$$
$$|T| = \sum\{r(\alpha) / \alpha \text{ is undecidable in T}\} \quad ;$$
$$T \leq T' \iff T' \subseteq T$$

then $(\mathcal{T},\leq,\delta,|\ |)$ is a p-p-m-space.

The number $|T|$ represents, in a sense, the amount of (relevant) informations contained T; indeed, if $|T|=0$ then all the relevant sentences are decidable in T while if $|T|=1$ then no relevant sentence is decidable in T. Likewise $\delta(T,T')$ is a measure of the contrast between T and T' and if $\delta(T,T')=0$ ($\delta(T;T')=1$) then T and T' agree (disagree, respectively) in all the relevant sentences. In Gerla [1991]b the p-p-m-spaces so obtained are used in connection with Popper's verisimilitude question.

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