t-norm based product of LR fuzzy numbers

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Abstract: The aim of this paper is to provide new results regarding the effective practical computation of t-norm-based multiplication of LR fuzzy numbers.

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1. Introduction

The present paper is devoted to the derivation of a fast computation (approximate) formula for t-norm-based multiplication of LR fuzzy numbers. The multipliciation rule of LR fuzzy numbers is well known in the case of "min"-norm [1]. An important feature of the approach by t-norms is that it allows control the growth of spreads i.e. growth of uncertainty in calculations. Using different t-norms one can choose different aggregation concepts for particular problems between the extremes of the most pessimistic (min-norm) and the most optimistic (the weakest t-norm T_W) approaches.

We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy sets of the real line and by $\mathcal{FN}_{LR}(\mathbb{R})$ the set of all fuzzy numbers of LR-type:

$$\tilde{a}(x) = \begin{cases} L\left(\dfrac{a-x}{\alpha_1}\right) & \text{if } x \in [a-\alpha_1, a] \\ R\left(\dfrac{x-a}{\alpha_2}\right) & \text{if } x \in [a, a+\alpha_2] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where a is the mean value of $\tilde{a}$; L and $R : [0,1] \to [0,1]$ are the shape (reference) functions of $\tilde{a}$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, which are non-increasing, continuous

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mappings; α_1 is the left spread and α_2 is the right spread of $\tilde{a}$. We use the notation $\tilde{a} = (a, \alpha_1, \alpha_2)_{LR}$.

For the sake of simplicity we consider in the following only positive fuzzy sets, i.e. $\tilde{a} \in \mathcal{F}(\mathbb{R})$ such that $\tilde{a}(x) = 0 \; \forall x < 0$. Denote the set of positive fuzzy sets by $\mathcal{F}(\mathbb{R}^+)$ and the set of positive LR-type fuzzy numbers by $\mathcal{FN}_{LR}(\mathbb{R}^+)$.

It is well known that a classical real operation $* : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ can be fuzzified via Zadeh's extension principle to an $\mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$ operation. Taking $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ we have

$$\tilde{a} * \tilde{b}(z) = \sup_{x * y = z} T\left(\tilde{a}(x), \tilde{b}(y)\right) \tag{2}$$

where $T : [0,1] \times [0,1] \rightarrow [0,1]$ is an arbitrary t-norm (Zadeh proposed $T = \min$).

A review of the multiplication of fuzzy numbers of LR-type in the case of "min"-norm can be found in Dubois, Prade [1]. Let us consider two positive fuzzy numbers $\tilde{a} = (a, \alpha_1, \alpha_2)$, $\tilde{b} = (b, \beta_1, \beta_2) \in \mathcal{FN}_{LR}(\mathbb{R}^+)$ of the same shape. In contrary to the sum $\tilde{a} \oplus \tilde{b}$, the product $\tilde{a} \odot \tilde{b}$ will not be an LR-type fuzzy number of the same form. To determine the membership degree $(\tilde{a} \odot \tilde{b})(z)$ without any approximation a second-order equation has to be (and can be) solved for every z. Nevertheless, if the spreads of $\tilde{a}$ and $\tilde{b}$ are small compared with their mean values, i.e. if $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$ then the terms like $O(\frac{\alpha_1 \beta_1}{ab})$ can be neglected. It leads to the following approximation formula

$$(a, \alpha_1, \alpha_2)_{LR} \odot (b, \beta_1, \beta_2)_{LR} \simeq (ab, a\beta_1 + b\alpha_1, a\beta_2 + b\alpha_2)_{LR} \tag{3}$$

that is, in the case of "min"-norm $\tilde{a} \odot \tilde{b}$ is approximately an LR-type fuzzy number described by the same shape functions L and R.

Using in (2) an arbitrary t-norm the situation becomes much more complicated even for the addition. Recall that the mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm iff it is non-decreasing in each argument and $T(x, 1) = x$, $\forall x \in [0,1]$. A t-norm T is said to be Archimedean iff T is continuous and $T(x, x) < x$, $\forall x \in]0,1[$. Every Archimedean t-norm

T is representable by a continuous and decreasing function $g : [0,1] \to [0,\infty[$ with $g(1) = 0$ and

$$T(x,y) = g^{[-1]}\big(g(x) + g(y)\big) \tag{4}$$

where $g^{[-1]}$ is the pseudo-inverse of g, defined by

$$g^{[-1]}(y) = \begin{cases} g^{-1}(y) & \text{if } y \in [0, g(0)] \\ 0 & \text{if } y \in [g(0), \infty] \end{cases} \tag{5}$$

Function g is called the additive generator of T.

There exist a lot of papers in the literature on t-norm based addition of fuzzy numbers, but only few ones concerning the multiplication. The (g,p)-fuzzification concept of Kovács [3,4] is based on the idea that dealing with a particular problem the membership functions and the t-norm in (2) should not be modelled undependently from each other. Kovács investigated the extended addition of LR fuzzy numbers with shape functions $L = R = g^{-1}$ under t-norm T_p having additive generator g^p, $p \geq 1$. It was shown that if $\tilde{a}, \tilde{b} \in \mathcal{FN}_{g^{-1}g^{-1}}(\mathbb{R})$ and $\tilde{a} = \big(a, \alpha_1, \alpha_2\big)_{g^{-1}g^{-1}}$, $\tilde{b} = \big(a, \beta_1, \beta_2\big)_{g^{-1}g^{-1}}$ then $\tilde{c} = \tilde{a} \oplus \tilde{b}$ defined by (2) with $T = T_p$ is also a fuzzy numbers with the same shape functions $L = R = g^{-1}$, i.e. $\tilde{c} \in \mathcal{FN}_{g^{-1}g^{-1}}(\mathbb{R})$ and

$$\big(a, \alpha_1, \alpha_2\big)_{g^{-1}g^{-1}} \oplus \big(b, \beta_1, \beta_2\big)_{g^{-1}g^{-1}} = \big(a + b, \gamma_1(p), \gamma_2(p)\big)_{g^{-1}g^{-1}} \tag{6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \gamma_1(p) &= \big\|(\alpha_1, \beta_1)\big\|_q = \sqrt[q]{(\alpha_1)^q + (\beta_1)^q} \\ \gamma_2(p) &= \big\|(\alpha_2, \beta_2)\big\|_q = \sqrt[q]{(\alpha_2)^q + (\beta_2)^q} \end{aligned} \tag{7}$$

For brevity let us write in the following just g instead of $g^{-1}g^{-1}$ using notations like $\mathcal{FN}_g(\mathbb{R})$ or $\tilde{a} = \big(a, \alpha_1, \alpha_2\big)_g$.

2. Results

We will deduce a corresponding formula for multiplication of LR fuzzy numbers. But, on the analogy of the min-based operations [1], we will neglect some quadratic terms and infer an approximation formula in the sense of (3).

Proposition. If the multiplication is extended via t-norm T_p having additive generator $g^p : [0,1] \to [0,\infty[$ $(p \geq 1)$ then for the extended product of two positive fuzzy numbers $\tilde{a} = \left(a, \alpha_1, \alpha_2\right)_g$, $\tilde{b} = \left(b, \beta_1, \beta_2\right)_g \in \mathcal{FN}_g(\mathbb{R}^+)$ we get the following approximation formula:

$$\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g \simeq \left(c, \gamma_1(p), \gamma_2(p)\right)_g \tag{8}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $c = a \cdot b$ and

$$\begin{aligned} \gamma_1(p) &= \left\|(b\alpha_1, a\beta_1)\right\|_q = \sqrt[q]{(b\alpha_1)^q + (a\beta_1)^q} \\ \gamma_2(p) &= \left\|(b\alpha_2, a\beta_2)\right\|_q = \sqrt[q]{(b\alpha_2)^q + (a\beta_2)^q} \end{aligned} \tag{9}$$

provided that the spreads of $\tilde{a}$ and $\tilde{b}$ are small compared with their mean values, i.e. $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$. That is, the fuzzy number $\tilde{c} = \left(c, \gamma_1(p), \gamma_2(p)\right)_g \simeq \tilde{a} \odot \tilde{b}$ is described by the same shape functions $L = R = g^{-1}$ as $\tilde{a}$ and $\tilde{b}$.

This proposition can easily be proved applying results of Fullér, Keresztfalvi [2] and the following fact originally used by Kovács [3,4]: there exists a natural bijection between the normed linear space $\mathbb{R}^n$ and the normed linear space of linear functionals on $\mathbb{R}^n$, i.e. $Lin(\mathbb{R}^n, \mathbb{R})$. Moreover, this bijection is an isometry if we normalize $Lin(\mathbb{R}^n, \mathbb{R})$ with the $\|\cdot\|_p$ norm and $\mathbb{R}$ with the $\|\cdot\|_q$ norm where $\frac{1}{p} + \frac{1}{q} = 1$.

3. Remarks

We first make some simple remarks.

(i) If $\tilde{a} \in \mathcal{FN}_g(\mathbb{R}^-)$ and $\tilde{b} \in \mathcal{FN}_g(\mathbb{R}^+)$ (10) turns into

$$\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g \simeq \left(a \cdot b, \left\|(b\alpha_1, a\beta_2)\right\|_q, \left\|(b\alpha_2, a\beta_1)\right\|_q\right)_g \tag{11}$$

(*ii*) If $\tilde{a} \in \mathcal{FN}_g(I\!R^-)$ and $\tilde{b} \in \mathcal{FN}_g(I\!R^-)$ (10) turns into

$$\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g \simeq \left(a \cdot b, \left\|(b\alpha_2, a\beta_2)\right\|_q, \left\|(b\alpha_1, a\beta_1)\right\|_q\right)_g \qquad (12)$$

The both extreme cases are also worth mentioning:

(*iii*) If $p = 1$ (and $q = \infty$) then T_p is equal to the well known Lukasiewicz's t-norm: $T_1(x, y) = T_L(x, y) = \max\{0, x + y - 1\}$. In this case $\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g$ has the smallest spreads

$$\begin{aligned} \gamma_1(1) &= \left\|(b\alpha_1, a\beta_1)\right\|_\infty = \max\left\{b\alpha_1, a\beta_1\right\} \qquad (13) \\ \gamma_2(1) &= \left\|(b\alpha_2, a\beta_2)\right\|_\infty = \max\left\{b\alpha_2, a\beta_2\right\} \end{aligned}$$

(*iv*) It is easy to see that $\lim_{p\to\infty} T_p(x, y) = \min\{x, y\}$. Thus, if $q = 1$ (and $p = \infty$), then we get back the min-based addition and $\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g$ has the greatest spreads:

$$\begin{aligned} \gamma_1(\infty) &= \left\|(b\alpha_1, a\beta_1)\right\|_1 = b\alpha_1 + a\beta_1 \qquad (14) \\ \gamma_2(\infty) &= \left\|(b\alpha_2, a\beta_2)\right\|_1 = b\alpha_2 + a\beta_2 \end{aligned}$$

(*v*) Obviously, (10) can be generalized to the case of multiplication of LR fuzzy intervals. Considering e.g. positive intervals $\tilde{a} = \left(a_1, a_2, \alpha_1, \alpha_2\right)_g$, $\left(b_1, b_2, \beta_1, \beta_2\right)_g \in \mathcal{FI}_g(I\!R^+)$, (10) becomes

$$\left(a_1, a_2, \alpha_1, \alpha_2\right)_g \odot \left(b_1, b_2, \beta_1, \beta_2\right)_g \simeq \left(a_1 b_1, a_2 b_2, \left\|(b_1\alpha_1, a_1\beta_1)\right\|_q, \left\|(b_2\alpha_2, a_2\beta_2)\right\|_q\right)_g \quad (15)$$

(*vi*) Let us write (10) in the form

$$\left(a, \alpha_1, \alpha_2\right)_g \odot \left(b, \beta_1, \beta_2\right)_g \simeq \left(c, \left\|\left(\frac{c\alpha_1}{a}, \frac{c\beta_1}{b}\right)\right\|_q, \left\|\left(\frac{c\alpha_2}{a}, \frac{c\beta_2}{b}\right)\right\|_q\right)_g \qquad (16)$$

where $c = a \cdot b$ of course. This form implies an obvious generalization to the case of more than two multiplicands. If $\tilde{a}_i = \left(a_i, \alpha_{1i}, \alpha_{2i}\right)_g \in \mathcal{FN}_g(I\!R^+)$ are n positive (for simplicity) LR fuzzy numbers, then (16) turns into

$$\left(a_1, \alpha_{11}, \alpha_{21}\right)_g \odot \left(a_2, \alpha_{12}, \alpha_{22}\right)_g \odot \ldots \odot \left(a_n, \alpha_{1n}, \alpha_{2n}\right)_g \simeq \left(c, \gamma_1(p), \gamma_2(p)\right)_g \quad (17)$$

with

$$\begin{aligned} c &= a_1 \cdot a_2 \cdot \ldots \cdot a_n \\ \gamma_1(p) &= \left\| \left(\frac{c\alpha_{11}}{a_1}, \frac{c\alpha_{12}}{a_2}, \ldots, \frac{c\alpha_{1n}}{a_n}\right) \right\|_q \\ \gamma_2(p) &= \left\| \left(\frac{c\alpha_{21}}{a_1}, \frac{c\alpha_{22}}{a_2}, \ldots, \frac{c\alpha_{2n}}{a_n}\right) \right\|_q \end{aligned}$$

References

[1] D. Dubois, H. Prade: Fuzzy Sets and Systems: Theory and Application, Academic Press, New York, (1980)

[2] R. Fullér, T. Keresztfalvi: On generalization of Nguyen's theorem, *Fuzzy Sets and Systems*, **41**, (1991)

[3] M. Kovács: Stable embedding of ill–posed linear equality and inequality systems into fuzzified systems, *Fuzzy Sets and Systems*, (to appear)

[4] M. Kovács: On the g–fuzzy linear systems, *BUSEFAL*, **37**, (1988) 69–77

[5] H. T. Nguyen: A note on the extension principle for fuzzy sets, *J. Math. Anal. Appl.*, **64**, No. 2. (1978) 369–380; vagy UCB/ERL Memo M–611. Univ. of California, Berkley (1976)