INCOMPATIBLE FUZZY LINEAR PROGRAMMING

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In the formulation of fuzzy linear programming proposed by Zimmermann, it is assumed that fuzzy goals and fuzzy constraints are explicitly given as fuzzy sets on oppropriate real lines respectively and are charecterized by membership functions in linear form. Under these assumptions the optimization is performed by using a standard linear programming technique . In this paper we will consider the case if the set of feasible solutions is empty. Such a problem we will call the incompatible fuzzy linear programming.

Let us consider the fuzzy linear problem [2] ,

$$\begin{aligned} c \cdot x &\lesssim z \\ A \cdot x &\lesssim b \ , \\ x \quad & 0 \ , \end{aligned} \qquad (1)$$

Where A is an m×n - matrix, c and x are the n-vectors, b is an m-vector.

Let

$$B = \begin{bmatrix} C \\ A \end{bmatrix} \qquad \text{and} \qquad d = \begin{bmatrix} z \\ b \end{bmatrix} \ .$$

Then the problem (1) we can write in the following form:

$$B\,x \leqslant d$$
$$x \geqslant 0 \qquad (2)$$

Now, let us introduce the functions $f_i(x), i = \overline{1,m+1}$ such that

$$f_i(x) = \begin{cases} 1 & \text{for} \quad (Bx)_i \leqslant d_i \,, \\ 1 - \dfrac{(Bx)_i - d_i}{p_i} & \text{for } d_i < (Bx)_i < d_i + p_i \,, \\ 0 & \text{for} \quad (Bx)_i \geqslant d_i + p_i \,. \end{cases}$$

The problem of the fuzzy mathematical programming is the following:

$$\sup_{x \geqslant 0} \; \min_{1 \leqslant i \leqslant m+1} f_i(x) \qquad (3)$$

An element $\bar{x}$ for which (3) takes place, is an optimal solution of fuzzy mathematical programming.

Negoita i Sularia [1] proved that the problem (3) is equivalent to the following linear programming problem:

$$\max_{a \in [0,1]} a \quad \text{subject to} \qquad (4)$$

$$Dy \leqslant g, \qquad (5)$$

$$y \geqslant 0, \qquad (6)$$

where

$$D = \begin{bmatrix} p_1 & c_1 & \cdots & c_n \\ p_2 & a_{11} & \cdots & c_{1n} \\ - & - & - & - \\ p_{m+1} & a_{m+1} & \cdots & a_{mn} \end{bmatrix}, \quad y = \begin{bmatrix} a \\ x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad g = \begin{bmatrix} d_1 + p_1 \\ \vdots \\ d_{m+1} + p_{m+1} \end{bmatrix}.$$

Now, let us assume, that the set of feasible solutions of the problem (4) is empty. Then the fuzzy linear programming problem we will call incompatible fuzzy linear programming problem.

A vector $y^0 = (y_1^0, y_2^0, \ldots, y_{n+1}^0)$ is called the best aproximate feasible solution of the incompatible fuzzy linear programming problem if

$$\sum_{i=1}^{m+1} \left(\sum_{k=1}^{n+1} d_{ik} y_k^0 - g_i \right)^2 = \min_{y \in R^{n+1}} \sum_{i=1}^{m+1} \left(\sum_{k=1}^{n+1} d_{ik} y_k - g_i \right)^2 \quad (7)$$

A vector y^0 which satisfies (4),(6) and (7) is called an optimal approximate sollution of the fuzzy linear programming problem.

Let

$$F(y_1, \ldots, y_{n+1}) = \sum_{i=1}^{m+1} \left(\sum_{k=1}^{n+1} d_{ik} y_k - g_i \right)^2 .$$

The above function can abtain minimum if

$$\frac{\partial F}{\partial y_k} = 0 \qquad k = \overline{1,n+1} .$$

Let d^j $(j = \overline{1,m+1})$ and $d_i (i = \overline{1,n+1})$ denote the rows and columns of the matrix D respectively. Then , the above equation we can write in the following form:

$$D^T \cdot D \cdot y = \begin{bmatrix} (d_1)^T \cdot g \\ \vdots \\ (d_{n+1})^T g \end{bmatrix} = h \qquad (8)$$

THEOREM 1. The equation (8) has at least one solution.

P r o o f. From the Theorem of Kronecker - Capelli it follows that the equation (8) has a solution iff $r[D^T, D] = r[D^T D | h]$. The matrix $[D^T, D]$ is an (n+1)-matrix. If the rank of this matrix is equal n+1 then the equation (8) has a unique solution. Now, let assume that $r[D^T, D] = s < n+1$. Then any set of s+1 rows of $[D^T, D]$ is dependent. Let us consider a set of s+1 rows of the matrix $[D^T, D | h]$

$$\sum_{j=1}^{m+1} (d_{j1})^2 \dots \sum_{j=1}^{m+1} d_{j1} a_{jm} \qquad \sum_{j=1}^{m+1} d_{j1} g_j$$

$$\text{---} \qquad (9)$$

$$\sum_{j=1}^{m+1} d_{j1} d_{j,s+1} \dots \sum_{j=1}^{m+1} d_{jm} d_{j,s+1} \qquad \sum_{j=1}^{m+1} d_{j,s+1} g_j$$

We multiply the first row by c_1, second by c_2, etc. s+1 by c_{s+1} and next sum up. In this way we abtain the following row

$$\sum_{j=1}^{m+1} \Big(\sum_{k=1}^{s+1} c_k d_{jk}\Big) d_{j1} \dots \sum_{j=1}^{m+1} \Big(\sum_{k=1}^{s+1} c_k d_{jk}\Big) a_{j,m+1} \sum_{j=1}^{m+1} \Big(\sum_{k=1}^{s+1} c_k d_{jk}\Big) g_j \qquad (10)$$

Because the set of s+1 columns of the matrix D is dependent them there exist the numbers $c_1, \dots, c_{s+1}$ not all zero such that

$$\sum_{k=1}^{s+1} c_k d_{jk} = 0.$$

So, the rows (9) are dependent i.e. the equation (8) has at least are solution.

Now, we will show that if y^0 satisfies equation (8) the function $F(y_1, \dots, y_{n+1})$ abtain the minimum in y^0.

THEOREM 2. If $y^0=(y_1^0,\dots,y_{n+1}^0)$ is a solution of(8) then for all (n+1)-vectors $y=(y_1,\dots,y_{n+1})$

$$F(y_1,\dots,y_{n+1}) \geqslant F(y_1^0,\dots,y_{n+1}^0).$$

P r o o f. Let us note that the function $F(y_1,\dots y_{n+1})$ we can write in the following form

$$F(y_1,\dots,y_{n+1}) = (Dy-g)^T(Dy-g)$$

Then

$$(Dy - g)^T(Dy - g) = y^TD^TDy - y^TD^Tg - g^TDy + g^Tg =$$
$$=(d^1y)^2+\dots+(d^{m+1}y)^2 - 2g_1d^1y-\dots-2g_{m+1}d^{m+1}\ y+g^Tg.$$

So, for $y=y^0$ we have

$$(Dy^0-g)^T\cdot(Dy^0-g) = (d^1y^0)^2+\dots+(d^{m+1}y^0)^2-2g_1d^1g^0-\dots-2g_{m+1}d^{m+1}y^0+$$
$$+g^Tg.$$

From the equation (8) we have,

$$y^TD^TDy^0 = y^Th = d^1g_1y+\dots+ d^{m+1}g_{m+1}\cdot y.$$

Moreover

$$y^TD^TD\cdot y^0 = d^1yd^1y^0 +\dots+d^{m+1}y\ d^{m+1}y^0.$$

So,

$$d^1g_1y +\dots+d^{m+1}g_{m+1}y = d^1yd^1y^0 +\dots+d^{m+1}yd^{m+1}y^0$$

and

$$F(y_1,\dots, y_{n+1}) = (d^1y)^2+\dots+(d^{m+1}y)^2- 2d^1yd^1y^0-\dots-2d^{m+1}yd^{m+1}y^0+$$
$$+g^Tg.$$

Analogously

$$F(y_1^0,\dots,y_{n+1}^0) = g^Tg - (d^1y^0)^2-\dots-(d^{m+1}y^0)^2.$$

From the above qualities it follows that

$$F(y_1,\dots y_{n+1}) - F(y_1^0,\dots,y_{n+1}^0) =$$
$$= (d^1\cdot y - d^1\cdot y^0)^2 +\dots+ (d^{m+1}y - d^{m+1}y^0)^2 \geqslant 0.$$

So, for any n-vector y

$$F(y_1,\dots,y_{n+1}) \geqslant F(y_1^0,\dots,y_{n+1}^0).$$

From the above theorems it follows that each solution of the equation (8) is the best approximate feasible solution of the incompatible fuzzy linear programming problem.

R E F E R E N C E S

1. Negoita C.V., Sulsaria M. On fuzzy mathematical programming and tolerances in planning. Econ. Cybern.Econ.Comput.Stud. Res. 1976 nr 1 ss. 3-14.
2. Zimmermann H.J. Optimization in fuzzy environment. Proc. TIMS / ORSA Conference. Puerto Rico 1974.