A REMARK ON FUZZY PREFERENCE MODELLING

Zsolt Borso

Central Research Institute for Physics

P.O.Box 49, Budapest 1525 Hungary

ABSTRACT. The problem of defining strict preference P, indifference I and incomparability J associated with a given fuzzy preference relation R is investigated. New expressions are obtained when R is strongly S-complet. The theory is illustrated by a multicriteria decision model.

Key words: Preference modelling, strongly S-complete relations, strict preference, indifference and incomparability relations, boundary conditions.

1. PREFERENCE MODELLING

Our starting point is an fuzzy relation R on the set of alternatives. That is, a function

$$R: A \times A \to [0,1]$$

such that for any $a,b \in A$ is the truth value of the statement "a is not worse than b". Then we are to define strict preference, indifference and incomparability as fuzzy relations.

We are searching for relations which satisfy certain reasonable conditions (Ovchinnikov and Roubens [1],[2]):

A1. For any two alternatives a,b the values of P(a,b), I(a,b) and J(a,b) depend only on R(a,b) and R(b,a), so there exist $p,i,j:[0,1]x[0,1]\to[0,1]$ functions such that

$$P(a,b) = p(R(a,b),R(b,a)),$$
$$I(a,b) = i(R(a,b),R(b,a)),$$
$$J(a,b) = j(R(a,b),R(b,a)),$$

A2. p(x,y) is nondecreasing in its first place and nonincreasing in its second place;

i(x,y) is nondecreasing with respect to both arguments;

j(x,y) is nonincreasing with respect to both arguments;

A3. P is antisymmetric, I and J are symmetric relations.

Let $T,S:[0,1]x[0,1] \to [0,1]$ and $n:[0,1] \to [0,1]$ be functions modelling intersection, union and complementation in fuzzy set theory, see Weber [4]. We introduce them following Lukasiewicz (see also Ovchinnikov and Roubens [1]):

$$T(x,y) = \phi^{-1}(\max\{ \phi(x) + \phi(y) -1), 0\})$$
$$S(a,b) = \phi^{-1}(\min\{ \phi(x) + \phi(y)), 1\})$$
$$n(x) = \phi^{-1}(1-\phi(x))$$

where ϕ is a strictly increasing continuous function from the unit interval onto itself satisfying boundary conditions $\phi(0)=0$ and $\phi(1)=1$.

Fodor [3] proposed the next definitions of strict preference, indifference and incomparability relations:

(1.1) $P(a,b) = T(R(a,b),n[R(b,a)]),$

(1.2) $I(a,b) = \min\{R(a,b),R(b,a)\},$

(1.3) $J(a,b) = \min\{n[R(a,b)],n[R(b,a)]\}.$

The (1.1)-(1.3) definitions fulfil the A1-A3 conditions. If we suppose that f(x)=x than R is strongly S-complete (i.e. S(R(a,b),R(b,a))=1 for any $a,b \in A$) if and only

$$R(a,b) + R(b,a) \geq 1,$$

or equivalently,

$$1-R(a,b) + 1-R(b,a) \leq 1.$$

This inequality implies that

$$\min\{ n(R(a,b)), n(R(b,a)) \} \leq 0.5 \text{ , i. e. } J(a,b)\leq 0.5 \text{ for any } a,b\in A.$$

In the next section we modify definitions (1.1)-(1.3) in order to have full range [0,1] for P, I, and J.

2. ANALYSIS

First we obtain boundary conditions for p, i, j.

Lemma . If R is strongly S-complete and fulfils A1-A3, then (2.1)-(2.9) hold:

(2.1) $p(1,0) = \max_{x,y}\{ p(x,y) \} = 1$

(2.2) $p(1,1) = \min_{x,y}\{ p(x,y) \} = 0$

(2.3) $p(0.5,0.5) = \min_{x,y}\{ p(x,y) \} = 0$

(2.4) $i(1,0) = \min_{x,y}\{ i(x,y) \} = 0$

(2.5) $i(1,1) = \max_{x,y}\{ i(x,y) \} = 1$

(2.6) $i(0.5,0.5) = \min_{x,y}\{ i(x,y) \} = 0$

(2.7) $j(1,0) = \min_{x,y}\{ j(x,y) \} = 0$

(2.8) $j(1,1) = \min_{x,y}\{ j(x,y) \} = 0$

(2.9) $j(0.5,0.5) = \max_{x,y}\{ j(x,y) \} = 1$

Proof: The lemma directly follows from the definitions.

From now we denote $x=R(a,b)$, $y=R(b,a)$ for short. Now we are looking for p, i, j in the next form:

$$\phi(p(x,y)) = p_1*\phi(x) + p_2*\phi(y) + p_3$$

$$\phi(i(x,y)) = i_1*\phi(x) + i_2*\phi(y) + i_3$$

$$\phi(j(x,y)) = j_1*\phi(x) + j_2*\phi(y) + j_3 .$$

if (x,y) is in the triangle defined by vertices (1,0), (1,1), (0.5,0.5). These functions are completely determined by values in points (1,0), (1,1), (0.5,0.5). Thus our lemma implies the following expressions:

$$\phi(p(x,y)) = \phi(x) - \phi(y),$$

$$\phi(i(x,y)) = \phi(x) + \phi(y) - 1$$

$$\phi(j(x,y)) = -2\phi(x) + 2,$$

or equivalently

(2.10) $P(a,b) = T(\ R(a,b),\ n(R(b,a))\ ),$

(2.11) $I(a,b) = T(\ R(a,b),\ R(b,a)\ )$ and

(2.12) $J(a,b) = \min\{\ \phi^{-1}(2-2\phi(R(a,b))),\ \phi^{-1}(2-2\phi(R(b,a)))\}$

which hold in the triangle with vertices (1,0), (1,1), (0,1).

Boundary conditions (2.1)-(2.9) can be expressed in the general case, when R is not strongly S-complete and $(x,y)\in[0,1]x[0,1]$. In this case the values of p, i and j have to be determined in points (0,0), (0,1), (1,1). The relations P, I, and J implied by those boundary conditions are the same that Fodor proposed, (1.1)-(1.3).

Finally an example is presented. Let A be the set of the alternatives, c_1, c_2, ...,c_k criteria with weights w_1, w_2, ...,w_k, representing their relative importances ($\Sigma w_i=1$), and $>_1$, $>_2$,...$>_k$; $=_1$, $=_2$,...$=_k$ complete orders on A. We will use $\phi(x)=x$ for short.

Let

$$w^{+}(a,b) = \Sigma_{\{i,\ a >_i b\}} w_i,$$

$$w^{=}(a,b) = \Sigma_{\{i,\ a =_i b\}} w_i \text{ and}$$

$$w^{-}(a,b) = \Sigma_{\{i,\ a <_i b\}} w_i.$$

It is clear in this case that

$$R(a,b) = w^{+}(a,b) + w^{=}(a,b).$$

Obviously R is strongly S-complet. Using (2.10)-(2.12),

$$P(a,b) = w^{+}(a,b) - w^{-}(a,b),$$

$$I(a,b) = w^{=}(a,b) \text{ and}$$

$$J(a,b) = 2w^{-}(a,b)$$

can be obtained if $w^{+}(a,b) \geq w^{-}(a,b)$.

REFERENCES

[1] Ovchinnikov, S., Roubens, M., On Strict Preference Relations, Fuzzy Sets and Systems (to appear).

[2] Ovchinnikov, S., Roubens, M., On Strict Preference, Indifference, and Incomparability Relations, Fuzzy Sets and Systems (to appear).

[3] Fodor, J. C., An Axiomatic Approach to Fuzzy Preference Modelling, submitted.

[4] Weber, S., A General Concept of Fuzzy Connectives, Negations and Implications Based on t-norms and t-conorms, Fuzzy Sets and Systems 11. (1983) 115-134.