THE CARDINALITY OF FUZZY SETS AND CONTINUUM HYPOTHESIS (I)

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ABSTRACT

Up to now, the study of the cardinality of fuzzy sets has hardly advanced since it is difficult to give it an appropriate definition. Alth ough D. Dubois and H. Prade had ever defined it in [1], it is with some harsh terms and is not very reasonable as we point out in this paper. In this paper, we give a general definition of the cardinality of fuzzy sets. Based on this definition, we not only obtain a large part of the results with repspect to the cardinality in Cartor's set theory, but also get many new properties of the cardinality of fuzzy sets. Especially, from the cardinality of fuzzy sets, we may gain some new idea on the continuum hypothesis which is a well-known problem.

KEYWARDS: Cardinality, F-cardinality, GCH

1. THE DEFINITION OF THE CARDINALITY OF FUZZY SETS AND ITS EQUIVALENT CONDITIONS

For any fuzzy set $A \in \mathcal{F}(X)$, $\forall\lambda\in[0, 1]$, the λ-cut and the strong λ-cut of A is respectively defined as the follqwing:

$$A_\lambda=\{x\in X \mid A(x)\geq\lambda\}, \qquad A_{\dot\lambda}=\{x\in X \mid A(x)>\lambda\}$$

Where $A(x)=\mu_A(x)$, which $A(x)$ is more convenient than $\mu_A(x)$.

Definition 1.1 The fuzzy set A is equipollent to the fuzzy set B if $(\forall\lambda\in[0, 1])(A_\lambda\sim B_\lambda)$, and we also write $A\sim B$. Fuzzy sets are partitioned by the equivalence relation and the equivalence class containing A is called the cardinality of A, denoted by $|A|$. For convenience we call the cardindlity of fuzzy sets F cardinality.

For example, let A and B respectively be a fuzzy set on [a, b]. If A and B are strictly monotone continuous functions and with $A(a)=B(c)$, $A(b)=B(d)$, then $A\sim B$.

Lemma1.1 Let A be a fuzzy set and $\{\alpha_k\}$ be a monotone sequence in (0, 1). We have

1) $\forall\lambda\in(o,\ 1]$, if $\alpha_k\nearrow\lambda$, then

$$\bigcap_{k=1}^{\infty} A_{\alpha_k}=A_\lambda\ ;\qquad \bigcap_{k=1}^{\infty} A_{\underset{\bullet}{\alpha_k}}=A_\lambda\ \ (\alpha_k<\lambda\ )$$

2) $\forall\lambda\in[0,\ 1)$, if $\alpha_k\searrow\lambda$ and $\alpha_k>\lambda$, then

$$\bigcup_{k=1}^{\infty} A_{\alpha_k}=A_{\underset{\bullet}{\lambda}}\ ;\qquad \bigcup_{k=1}^{\infty} A_{\underset{\bullet}{\alpha_k}}=A_{\underset{\bullet}{\lambda}}$$

Lemma 1.2 Let $\{A_n\},\{B_n\}$ be two sequences of sets which are not fuzzy sets. If they satisfy the condition: $(\forall n)(A_n\subseteq A_{n+1},\ B_n\subseteq B_{n+1})$, then $(\forall n)(A_n\sim B_n)\Longrightarrow \bigcup_{n=1}^{\infty} A_n\sim\bigcup_{n=1}^{\infty} B_n$.

We always assume that Q_0 is a countable dense subset of (0, 1).

Theorem1.1 $A\sim B\Longleftrightarrow(\forall\alpha\in Q_0)(A_\alpha\sim B_\alpha)$.

For any fuzzy set A, write $\text{supp}A=A_0=\{x\,|\,A(x)>0\}$, which is called the support of A.

Lemma1.3 For any fuzzy sets A and B, if there exists a bijection f: suppA$\longrightarrow$suppB such that $A(x)=B(f(x))$, then $A\sim B$.

Corollary. If f is a transformation(bijection) on suppA and put $A'(x)=A(f(x))$, then $A'\sim A$.

This means that F cardinality is fixed under any transformation.

Remark: The inversion of lemma 1.3 is not true.

For example, taking B=[0, 2] and A defined as

$$A(x)=\begin{cases} 1 & , x\in[0,\ 1] \\ 0.5 & , x\in(1,\ 2] \end{cases}$$

clearly $A\sim B$. But when $x\in(1,\ 2]$, for any bijection f: $\mathrm{supp}A \longrightarrow \mathrm{supp}B$, $A(x)\neq B(f(x))$. However, when the support is finite, the inversion is true.

Theorem 1.2 For any fuzzy sets A and B, if suppA and suppB are finite, then $A\sim B$ if and only if $|\mathrm{supp}A|=|\mathrm{supp}B|=n$ and there exists a n-ary permutation σ such that $A(x_i)=B(y_{\sigma(i)})$, $1\le i\le n$.

Remark: When fuzzy sets A and B have the finite supports, clearly $A\sim B \Longrightarrow |\mathrm{supp}A|=|\mathrm{supp}B|=n$ and $\sum_{i=1}^{n}A(x_i) = \sum_{j=1}^{n}B(y_j)$. But its inversion is not true. For example, let $A=(0.6,0.3)$, $B=(0.7,0.2)$. We have $A(x_1)+A(x_2)=B(y_1)+B(y_2)$, but $|A|\neq|B|$. In [1], the cardinality of fuzzy sets with finite supports is defined as $|A|=\sum_{i=1}^{n}A(x_i)$. This is weaker than the cardition of our definition. Also in [1], when suppA is infinite, $|A|$ is definied by $\sum_{i=1}^{\infty}A(x_i)$ if $\sum_{i=1}^{\infty}A(x_i)$ is convergent. Furthermore, it is extended as integral in [1]. Now we give a example to show that the definition in [1] is not very reasonable. Taking $A=[0,\ 2]$, $B=[0,\ 3]$, of course $A\sim B$. But by [1] we have

$$|A|=\int_0^2 A(x)dx < \int_0^3 B(y)dy = |B|$$

This is in disagreement with our elementary knowledge. However our definition contains case of Cantor's sets.

Theorem 1.3 $(\forall\lambda\in Q_0)(A_\lambda\sim B_\lambda)\Longrightarrow A\sim B$

Remark: The inversion of theorem 1.3 is not true. See following two fuzzy sets on real number field: given $a\in Q_0$,

$$A(x)=\begin{cases} a, & x\in Q \\ 0, & \text{otherwise} \end{cases} \quad ; \quad B(y)=\begin{cases} a-\frac{1}{y}, & y\in N \text{ and } y\geq n_0 \\ 0, & \text{otherwise} \end{cases}$$

where Q is the set of rational numbers, N is the set of natural numbers and n_0 satisfies the condition $a-\frac{1}{n_0} > 0$. Clearly A~B but $|A_a|=\aleph_0 > |B_a|=0$.

Corollary. $(\forall\lambda\in(0, 1])(A_\lambda \sim B_\lambda)\Longrightarrow A\sim B$

The mapping H: $[0, 1]\longrightarrow\mathcal{P}(X)$, $\lambda\longrightarrow H(\lambda)$, is called a set case on X if $(\forall\lambda_1,\lambda_2)(\lambda_1<\lambda_2\Longrightarrow H(\lambda_1)\supseteq H(\lambda_2))$. The set of the set cases on X is denoted by $\mathcal{U}(X)$.

In $\mathcal{U}(X)$, we define the operations $\cup, \cap, c$ as the following:

$$\bigcup_{t\in T} H_t: \quad (\bigcup_{t\in T} H_t)(\lambda) = \bigcup_{t\in T} H_t(\lambda)$$

$$\bigcap_{t\in T} H_t: \quad (\bigcap_{t\in T} H_t)(\lambda) = \bigcap_{t\in T} H_t(\lambda)$$

$$H^c: \quad H^c(\lambda) = (H(1-\lambda))^c$$

The representation theorem of fuzzy sets is as the following.

If we put the mapping

$$\phi: \mathcal{U}(X)\longrightarrow\mathcal{F}(X)$$
$$H\longrightarrow\phi(H) = \bigcup_{\lambda\in[0, 1]} \lambda H(\lambda)$$

then ϕ is an epimorphism from $(\mathcal{U}(X),\cup,\cap,c)$ to $(\mathcal{F}(X),\cup,\cap,c)$ and with

$$\phi(H)_{\underset{\bullet}{\lambda}} \subseteq H(\lambda) \subseteq \phi(H)_\lambda \quad , \quad \forall\lambda\in[0, 1]$$

$$\phi(H)_\lambda = \bigcap_{\alpha<\lambda} H(\alpha) \ , \quad \forall\lambda\in(0, 1]$$

$$\phi(H)_{\underset{\bullet}{\lambda}} = \bigcup_{\alpha>\lambda} H(\alpha) \ , \quad \forall\lambda\in[0, 1)$$

where λA is a fuzzy set with $(\lambda A)(x)=\lambda\wedge A(x)$.

In $\mathcal{U}(X)$, an equivalence relation ~ is defined as $(\forall H,H'\in\mathcal{U}(X))(H\sim H'\Longleftrightarrow\phi(H)=\phi(H'))$, and the equivalence class for H belonging to is denoted by [H]. So we get a quotient set $\mathcal{F}'(X)=\{[H] \mid H\in\mathcal{U}(X)\}$. If we define the operations $\cup, \cap, c$ in $\mathcal{F}'(X)$,

$$\{H_1\}\cup\{H_2\} = \{H_1\cup H_2\}$$

$$\{H_1\}\cap\{H_2\} = \{H_1\cap H_2\}$$

$$\{H^c\} = \{H\}^c$$

$$\bigcup_{t\in T} \{H_t\} = \{ \bigcup_{t\in T} H_t\}$$

$$\bigcap_{t\in T} \{H_t\} = \{ \bigcap_{t\in T} H_t\}$$

the mapping $\phi': \mathcal{F}'(X)\longrightarrow\mathcal{F}(X)$, $[H]\longrightarrow\phi'(H)=\phi(H)= \bigcup_{\lambda\in[0,\ 1]} \lambda H(\lambda)$, is an isomorphism. So $\{H\}$ can be regarded as a fuzzy set.

For $A\in\mathcal{F}(X)$, if the mapping H_A: $[0,\ 1]\longrightarrow\mathcal{P}(X)$ satisfies the condition: $(\forall\lambda\in[0,\ 1])(A_{\underset{\bullet}{\lambda}} \le H_A(\lambda) \le A_\lambda)$, the H_A is a set case because $\lambda_1 < \lambda_2 \Longrightarrow H_A(\lambda_1) \ge A_{\underset{\bullet}{\lambda_1}} \ge A_{\lambda_2} \ge H_A(\lambda_2)$. We call H_A a set case dissolved in A.

Lemma 1.4 Let H_A be a set case dissolved in A , and $\{\alpha_k\}$ be a monotone sequence in (0, 1), we have

1) for any $\lambda\in(0,\ 1]$, if $\alpha_k\nearrow\lambda$ and $\alpha_k<\lambda$, then $A_\lambda= \bigcap_{k=1}^{\infty} H(\alpha_k)$

2) for any $\lambda\in[0,\ 1)$, if $\alpha_k\searrow\lambda$ and $\alpha_k>\lambda$, then $A_{\underset{\bullet}{\lambda}}= \bigcup_{k=1}^{\infty} H(\alpha_k)$

THeorem 1.5 Let H_A , H_B respectively be the set case dissolved in A, B. Then $(\forall\lambda\in Q_0)(H_A(\lambda)\sim H_B(\lambda))\Longrightarrow A\sim B$.

Theorem 1.6 Let $\{A^{(n)}\}$, $\{B^{(n)}\}$ be respectively fuzzy sets sequence with $(n\ne m\Longrightarrow A^{(n)}\cap A^{(m)}=\phi$ and $B^{(n)}\cap B^{(m)}=\phi)$. Then$(\forall n)$ $(A^{(n)}\sim B^{(n)})\Longrightarrow \bigcup_{n=1}^{\infty} A^{(n)}\sim \bigcup_{n=1}^{\infty} B^{(n)}$.

2. THE COMPARISION OF F-CARDINALITY

Definition 2.1 Let $|A|=\alpha$, $|B|=\beta$. Define

$\alpha\le\beta$ iff $(\exists B'\subseteq B)(A\sim B')$; $\alpha<\beta$ iff $\alpha\le\beta$ and $\alpha\ne\beta$.

Monotonicity is clearly true: $A\subseteq B \Longrightarrow |A|\le|B|$.

Theorem 2.1 $|A|\le|B| \Longleftrightarrow (\forall\lambda\in[0,\ 1))(|A_{\underset{\cdot}{\lambda}}|\le|B_{\underset{\cdot}{\lambda}}|)$

Corollary 1. $|A|\le|B| \Longleftrightarrow (\forall\lambda\in Q_0)(|A_{\underset{\cdot}{\lambda}}|\le|B_{\underset{\cdot}{\lambda}}|)$

Corollary 2. $|A|<|B| \Longleftrightarrow |A|\le|B|$ and $(\exists\lambda\in[0,\ 1))(|A_{\underset{\cdot}{\lambda}}|<|B_{\underset{\cdot}{\lambda}}|)$

Corollary 3. $|A|<|B|\le|C|$ or $|A|\le|B|<|C| \Longrightarrow |A|<|C|$.

Theorem 2.2 (The generalized form of Cantor-Schröder-Bernstein's theorem): $|A|\le|B|$ and $|B|\le|A| \Longrightarrow |A|=|B|$.

Corollary 1. $A\subseteq B\subseteq C$ and $A\sim C \Longrightarrow B\sim C$.

Corollary 2. The relation "$\le$" for F-cardinality is a partial order.

Remark: The relation "$\le$" for F-cardinality is not a linear order, namely there exist incomparable F-cardinalities. For example, taking $A=(0.1,\ 0.4)$, $B=(0.2,\ 0.3)$, when $\lambda\in[0.1,\ 0.2)$, $|A_{\underset{\cdot}{\lambda}}|=1<|B_{\underset{\cdot}{\lambda}}|=2$; but when $\lambda\in[0.3,\ 0.4)$, $|A_{\underset{\cdot}{\lambda}}|=1>|B_{\underset{\cdot}{\lambda}}|=0$. So $|A|$ and $|B|$ are incomparable.

Corollary 3. Between any F-cardinalities α and β, there are four relations: $\alpha=\beta$, $\alpha<\beta$, $\alpha>\beta$, α and β are incomparable. And any two relations among them are not at the same time true.

Now we consider such kind of fuzzy sets: $A(x)=t$, $t\in[0,\ 1]$, called constant fuzzy sets. Of course, a constant fuzzy set can not always compare with another. For example, $A=(0.1,0.1,0.1)$, $B=(0.2,0.2)$. When $\lambda\in[0,\ 0.1)$, $|A_{\underset{\cdot}{\lambda}}|=3>2=|B_{\underset{\cdot}{\lambda}}|$; when $\lambda\in[0.1,\ 0.2)$,

$|A_{\lambda}|=0<2=|B_{\lambda}|$. So $|A|$ can not compare with $|B|$.

Clearly, there is the fact: Let A, B are constant fuzzy sets. If $|suppA|=|suppB|$, then $|A|$ can compare with $|B|$. On the contary, it is not true . For example, if A=(0.2, 0.2), B=(0.2, 0.2, 0.2), then $|A|<|B|$, but $|suppA|\neq|suppB|$.

It is common knowledge that the cardinality of a finite ordinary set can be expressed by a natural number that the quantity of elements in the set. For a fuzzy set, although its support is finite, its cardinality be expressed by natural numbres even real numbers, because the relation of big or small for F-cardinality is only partial order, except for constant fuzzy sets under certain condition.

Lemma 2.1. If $A(x)\equiv t$, $B(y)\equiv s$ are two constant fuzzy sets, then $|A|=|B|\Longleftrightarrow t=s$ and $|suppA|=|suppB|$.

So we can measure the cardinality of a fuzzy set based on two quanlity: $|suppA|$ and the grade of membership. When $A(x)\equiv t$ is a constant fuzzy set, we denote $<|suppA|,t>:=|A|$. As a result, the cardinality of such kind of fuzzy sets has more concrete expression. Especially, when $|suppA|=n$, $|A|=<n,t>$. Namely, the cardinality of a constant fuzzy set with finite support can be expressed by quality; of course it is two-demensional quantity. If $<n,t>$ is regarded as an ordered pair, then $<n,t>\in N\times[0, 1]$. Thereupon, the cardinalities of finite constant fuzzy sets are a total of $|N\times[0, 1]|=\aleph_0 2^{\aleph_0}=2^{\aleph_0}$. Generally we have

Theorem 2.3 Given universe $X\neq\phi$, when $|X|\leq 2^{\aleph_0}$, the cardinalities of the constant fuzzy sets on X are a total of $2^{\aleph_0}$; when $|X|>2^{\aleph_0}$, a total of $|X|$.

(to be continued)

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