Fuzzy Ideals Generated by Fuzzy Sets in Semigroups

Mo Zhi-wen and Wang Xue-ping

Department of Mathematics

Sichuan Normal University

Chengdu, 610066

P. R. China

Abstract

In this paper, the definition of a fuzzy left (right) ideal (a fuzzy ideal, a fuzzy bi-ideal, a fuzzy interior ideal) generated by a fuzzy set in semigroups is given. And the depiction of them are researched.

Keywords: Fuzzy left (right) ideal (Fuzzy ideal, Fuzzy bi-ideal, Fuzzy interior ideal) generated by a fuzzy set.

1. Introduction

The concept of fuzzy set, introduced in L. A. Zadeh [8], was applied to the elementary theory of groupoids and groups in A. Rosenfeld [7], semigroups in N. Kuroki [2 —— 5] and in authors' paper [6]. In the present paper we shall give the definition of a fuzzy left (right) ideal (a fuzzy ideal, a fuzzy bi-ideal, a fuzzy interior ideal) generated by a fuzzy set in semigroups. And some characterizations of them are showed.

2. Preliminaries

In this paper, we always suppose that S is a semigroup, S^1 a semigroup with unity element 1 (see [1]) and "F- " stands for "fuzzy".

A map f from S^1 to the closed interval [0, 1] is called a fuzzy set in S^1. Let $F(S^1)$ denote the set of all fuzzy sets in S^1. For any A, $B \in F(S^1)$, $A \subseteq B$ if and only if $A(x) \leq B(x)$, in the ordering of $[0, 1]$, for all $x \in S^1$.

Remark. For all $x \in S^1$, $A_i \in F(S^1)$, $i \in I$ (indexing set),

$$(\bigcup_i A_i)(x) = \bigvee_i A_i(x) = \sup_i A_i(x), \quad (\bigcap_i A_i)(x) = \bigwedge_i A_i(x) = \inf_i A_i(x).$$

Pu and Liu gave the definition of a fuzzy point (cf. [9], Definition 2.1 and Definition 2.2). Clearly that every $A \in F(S^1)$,

$$A = \bigcup_{x_\lambda \in A} x_\lambda$$

where $0 < \lambda \leq 1$, $x_\lambda \in A$ if and only if $x_\lambda \subseteq A$,

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{if } y \neq x; \end{cases}$$

for all $y \in S^1$. For x_λ, $y_\mu \in F(S^1)$, $x_\lambda \subseteq y_\mu$ if and only if $x = y$ and $\lambda \leq \mu$.

Definition 2.1. $f \in F(S^1)$ is called a fuzzy subsemigroup of S^1 if

$$f(xy) \geq \min\{f(x), f(y)\}$$

for all $x, y \in S^1$.

The definition of a fuzzy left (right) ideal (a fuzzy ideal, a fuzzy bi-ideal, a fuzzy interior ideal) of S^1 can be seen in [2 — 5].

Definition 2.2. [6]. Let $f \in F(S^1)$, the smallest fuzzy left (right, two-sided) ideal of S^1 containing f is called a fuzzy left (right, two-sided) ideal of S^1 generated by f, denoted by $\langle f \rangle$ ($[f]$, (f)) (where a fuzzy two-sided ideal means a fuzzy ideal).

Proposition 2.3. A fuzzy ideal of S^1 is a fuzzy bi-ideal of S^1, and also a fuzzy interior ideal of S^1.

Proof. Let $f \in F(S^1)$ be a fuzzy ideal of S^1. Then by definition of a fuzzy ideal, for all $x, y \in S^1$, $f(xy) \geq f(x)$, $f(xy) \geq f(y)$. This follows that

$$f(xy) \geq \min\{f(x), f(y)\}.$$

So f is a fuzzy subsemigroup of S^1. For $x, y, z \in S^1$, $f(xyz) = f((xy)z) \geq f(z)$

and $f(xyz) = f(x(yz)) \geqslant f(x)$. Thus

$$f(xyz) \geqslant \min\{f(x),\ f(z)\}.$$

This means f is a fuzzy bi-ideal of S^1. Now we further prove f is also a fuzzy interior ideal of S^1. For $x, a, y \in S^1$,

$$f(xay) \geqslant f(ay) \geqslant f(a).$$

This completes the proof.

Proposition 2.4. Let $f \in F(S^1)$, then f is a fuzzy ideal of S^1 if and only if f is a fuzzy interior ideal of S^1.

Proof. By Proposition 2.3, the necessity is obviously. Sufficiency. For $x, y \in S^1$, $f(xy) = f(xy1) \geqslant f(y)$, $f(xy) = f(1xy) \geqslant f(x)$, this means f is also a fuzzy ideal of S^1.

3. Fuzzy ideals generated by fuzzy sets

Theorem 3.1. Let $f \in F(S^1)$, then $\langle f \rangle = J$, where

$$J(a) = \sup_{\substack{a = x_1x_2 \\ x_1,\ x_2 \in S^1}} f(x_2)$$

for all $a \in S^1$.

Proof. For $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2} f(x_2) \geqslant \sup_{a = 1a} f(a) = f(a),$$

that is $J \geqq f$. For $x, y \in S^1$, by the defining way of J we have

$$J(xy) = \sup_{xy = x_1x_2} f(x_2) \geqslant \sup_{\substack{xy = (xz_1)z_2 \\ y = z_1z_2}} f(z_2) \geqslant \sup_{y = z_1z_2} f(z_2) = J(y).$$

This means J is a fuzzy left ideal of S^1. If I is a fuzzy left ideal of S^1 and $f \subseteq I$, then for $a \in S^1$, $f(a) \leq I(a)$, and

$$J(a) = \sup_{a = x_1x_2} f(x_2) \leq \sup_{a = x_1x_2} I(x_2) \leq \sup_{a = x_1x_2} I(x_1x_2) = I(a).$$

This follows that $J \subseteq I$. By Definition 2.2 we have that $\langle f \rangle = J$.

Theorem 3.2. Let $f \in F(S^1)$, then $[f] = J$, where

$$J(a) = \sup_{\substack{a = x_1x_2 \\ x_1,\, x_2 \in S^1}} f(x_1)$$

for all $a \in S^1$.

Proof. For $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2} f(x_1) \geq \sup_{a = a1} f(a) = f(a),$$

that is $J \supseteq f$. For $x, y \in S^1$,

$$J(xy) = \sup_{xy = x_1x_2} f(x_1x_2) \geq \sup_{\substack{xy = z_1(z_2y) \\ x = z_1z_2}} f(z_1) = \sup_{x = z_1z_2} f(z_1) = J(x).$$

This reaches J is a fuzzy right ideal of S^1. If I is a fuzzy right ideal of S^1 and $f \subseteq I$, then for $a \in S^1$, $f(a) \leq I(a)$, further more

$$J(a) = \sup_{a = x_1x_2} f(x_1) \leq \sup_{a = x_1x_2} I(x_1) \leq \sup_{a = x_1x_2} I(x_1x_2) = I(a),$$

thus $J \subseteq I$. By Definition 2.2 we have $[f] = J$.

Theorem 3.3. Let $f \in F(S^1)$, then

(i) $[\langle f \rangle] = (f)$;

(ii) $\langle [f] \rangle = (f)$;

(iii) $[\langle f \rangle] = \langle [f] \rangle$.

Proof. Firstly. By Theorem 3.2 we have $[\langle f \rangle]$ is a fuzzy right ideal of S^1. Secondly. For $x, y \in S^1$,

$$[\langle f \rangle](xy) = \sup_{xy = x_1x_2} \langle f \rangle (x_1) = \sup_{xy = x_1x_2} \sup_{x_1 = z_1z_2} f(z_2)$$

and

$$[\langle f \rangle](y) = \sup_{y = y_1y_2} \langle f \rangle (y_1) = \sup_{y = y_1y_2} \sup_{y_1 = w_1w_2} f(w_2).$$

Obviously $[\langle f \rangle](xy) \geq [\langle f \rangle](y)$, that is $[\langle f \rangle]$ is also a fuzzy left ideal of S^1. So $[\langle f \rangle]$ is a fuzzy ideal of S^1. Since $f \subseteq \langle f \rangle \subseteq [\langle f \rangle]$, we have $[\langle f \rangle] \supseteq f$. If I is a fuzzy ideal of S^1 and $I \supseteq f$, then by I is a fuzzy left ideal of S^1, we have $I \geq \langle f \rangle$, further more by I is a fuzzy right ideal of S^1, $I \geq [\langle f \rangle]$. Then by Definition 2.2, $[\langle f \rangle]$ is a fuzzy ideal of S^1 generated by f, that is $[\langle f \rangle] = (f)$. Similarly we can prove $\langle [f] \rangle = (f)$. So $[\langle f \rangle] = (f) = \langle [f] \rangle$.

By Theorem 3.1, 3.2 and 3.3 respectively, we can examine the following two propositions.

Proposition 3.4 [6]. Let $x_\lambda \in F(S^1)$, then $\langle x_\lambda \rangle = J$ ($[x_\lambda] = J$), where

$$J(a) = \begin{cases} \lambda & \text{if there exists } b \in S^1 \text{ such that } a = bx \ (a = xb), \\ 0 & \text{otherwise;} \end{cases}$$

for all $a \in S^1$.

Proposition 3.5 [6]. Let $x_\lambda \in F(S^1)$, then $(x_\lambda) = J$, where

$$J(a) = \begin{cases} \lambda & \text{if there exist } b,\ c \in S^1 \text{ such that } a = bxc, \\ 0 & \text{otherwise;} \end{cases}$$

for all $a \in S^1$.

4. Fuzzy interior ideals generated by fuzzy sets

Definition 4.1. Let $f \in F(S^1)$, the smallest fuzzy interior ideal of S^1 containing f is called a fuzzy interior ideal of S^1 generated by f, denoted by $\langle f \rangle_I$.

Theorem 4.2. Let $f \in F(S^1)$, then $\langle f \rangle_I = J$, where

$$J(a) = \sup_{\substack{a = x_1x_2x_3 \\ x_1,\ x_2,\ x_3 \in S^1}} f(x_2)$$

for all $a \in S^1$.

Proof. For $x, y \in S^1$,

$$J(xy) = \sup_{xy = x_1x_2x_3} f(x_2) \geqslant \sup_{\substack{xy = z_1z_2(z_3y) \\ x = z_1z_2z_3}} f(z_2) = J(x)$$

and

$$J(xy) = \sup_{xy = x_1x_2x_3} f(x_2) \geqslant \sup_{\substack{xy = (xw_1)w_2w_3 \\ y = w_1w_2w_3}} f(w_2) = J(y),$$

thus $J(xy) \geqslant \min\{J(x), J(y)\}$, that is J is a fuzzy subsemigroup of S^1. For $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2x_3} f(x_2) \geqslant \sup_{a = 1a1} f(a) = f(a),$$

hence $J \geq f$. For $x, a, y \in S^1$,

$$J(xay) = \sup_{xay = x_1x_2x_3} f(x_2) \geqslant \sup_{\substack{xay = (xz_1)z_2(z_3y) \\ a = z_1z_2z_3}} f(z_2) = J(a).$$

This follows that J is a fuzzy interior ideal of S^1. Let I be any fuzzy interior ideal of S^1 and $I \supseteq f$, then for all $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2x_3} f(x_2) \leqslant \sup_{a = x_1x_2x_3} I(x_2) \leqslant \sup_{a = x_1x_2x_3} I(x_1x_2x_3) = I(a).$$

By Definition 4.1, we get $\langle f \rangle_I = J$.

From Proposition 2.4, we have:

Proposition 4.3. Let $x_\lambda \in F(S^1)$, then $\langle x_\lambda \rangle_I = J$, where for all $a \in S^1$

$$J(a) = \begin{cases} \lambda & \text{if there exist } x_1, x_2 \in S^1 \text{ such that } a = x_1xx_2, \\ 0 & \text{otherwise;} \end{cases}$$

Set $f = x_\lambda$, then from Theorem 4.2 we also can prove Proposition 4.3.

5. Fuzzy bi-ideals generated by fuzzy sets

Definition 5.1. Let $f \in F(S^1)$, the smallest fuzzy bi-ideal of S^1 containing f is called a fuzzy bi-ideal of S^1 generated by f, denoted by $\langle f \rangle_B$.

Definition 5.2. Let $f \in F(S^1)$ be a fuzzy subsemigroup of S^1. f is called a fuzzy submonoid of S^1 if $f(1) \geqslant f(x)$ for all $x \in S^1$.

Theorem 5.3. Let $f \in F(S^1)$ be a fuzzy submonoid of S^1, the- $\langle f \rangle_B = J$, where

$$J(a) = \sup_{a = x_1x_2x_3} \min\{f(x_1),\ f(x_3)\}$$

for all $a \in S^1$.

Proof. For $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2x_3} \min\{f(x_1),\ f(x_3)\} \geqslant \sup_{a = 1\,1\,a} \min\{f(1),\ f(a)\} = \sup_{a = 1\,1\,a} f(a) = f(a),$$

that is $J \geq f$. For $x, y, z \in S^1$,

$$J(x) = \sup_{x = x_1x_2x_3} \min\{f(x_1),\ f(x_3)\},\quad J(z) = \sup_{z = z_1z_2z_3} \min\{f(z_1),\ f(z_3)\}$$

and

$$J(xyz) = \sup_{xyz = w_1w_2w_3} \min\{f(w_1),\ f(w_3)\} \geqslant \sup_{\substack{xyz = x_1(x_2x_3yz_1z_2)z_3 \\ x = x_1x_2x_3 \\ z = z_1z_2z_3}} \min\{f(x_1),\ f(z_3)\}.$$

But

$$\min\{J(x),\ J(z)\} = \sup_{\substack{x = x_1x_2x_3 \\ z = z_1z_2z_3}} \min\{\min\{f(x_1),\ f(x_3)\},\ \min\{f(z_1),\ f(z_3)\}\},$$

so $J(xyz) \geqslant \min\{J(x),\ J(z)\}$. Meanwhile if let $y = 1$, then we have

$$J(xz) \geqslant \min\{J(x),\ J(z)\}$$

for all $x, z \in S^1$, that is J is a fuzzy subsemigroup of S^1. Then J is a fuzzy bi-ideal of S^1. Let I be a fuzzy bi-ideal of S^1 and $I \supseteq f$, then for all $a \in S^1$,

$$J(a) = \sup_{a = x_1x_2x_3} \min\{f(x_1),\ f(x_2)\} \leqslant \sup_{a = x_1x_2x_3} \min\{I(x_1),\ I(x_2)\}$$

$$\leqslant \sup_{a = x_1x_2x_3} \min I(x_1x_2x_3) = I(a),$$

thus $J \subseteq I$. Hence $\langle f \rangle_B = J$.

Theorem 5.4. Let $f \in F(S^1)$. If S^1 is a regular semigroup, then $\langle f \rangle_B = J$, where

$$J(a) = \sup_{\substack{a = x_1x_2x_3 \\ x_1,\, x_2,\, x_3 \in S^1}} \min\{f(x_1),\ f(x_3)\}$$

for all $a \in S^1$.

Proof. From the proof of Theorem 5.3, we only need to prove $J(a) \geqslant f(a)$ for all $a \in S^1$. Indeed,

$$J(a) = \sup_{a = x_1x_2x_3} \min\{f(x_1),\ f(x_3)\} \geqslant \sup_{a = axa} \min\{f(a),\ f(a)\} = f(a)$$

for all $a \in S^1$.

Let $f = x_\lambda$ and by Theorem 5.4, we can obtain:

Theorem 5.5. Let $x_\lambda \in F(S^1)$. If S^1 is a regular semigroup, then $\langle x_\lambda \rangle_B = J$, where

$$J(a) = \begin{cases} \lambda & \text{if there exists } y \in S^1 \text{ such that } a = yxy, \\ 0 & \text{otherwise;} \end{cases}$$

for all $a \in S^1$.

Conversely, let B be a fuzzy regular open set in X_2 and p be a fuzzy point in $f^{-1}(B)$. Then $p \leqslant f^{-1}(B)$, i.e., $f(p) \leqslant B$. From hypothesis there exists a fuzzy semiopen set A in X_1 such that $p \leqslant A$ and $f(A) \leqslant B$, hence

$$p \leqslant A \leqslant f^{-1}f(A) \leqslant f^{-1}(B)$$

and

$$p \leqslant A = A_0 \leqslant (f^{-1}(B))_0.$$

Since p is arbitrary and $f^{-1}(B)$ is the union of all fuzzy points in $f^{-1}(B)$, $f^{-1}(B) \leqslant (f^{-1}(B))_0$, i.e., $f^{-1}(B) = (f^{-1}(B))_0$. Thus f is fuzzy almost semicontinuous.

Theorem 3. Let $f:(X_1, \delta_1) \to (X_2, \delta_2)$ be a mapping from a fuzzy space X_1 to a fuzzy regular space [2] X_2. Then f is fuzzy almost semicontinuous iff f is fuzzy semicontinuous.

Theorem 4. Let $f:(X_1, \delta_1) \to (X_2, \delta_2)$ be a mapping. Then the following are equivalent:

(1) f is fuzzy almost semiopen.

(2) $f(A) \leqslant (f(A^{-0}))_0$, for each $A \in \delta_1$.

(3) For each fuzzy set B of X_2 and each fuzzy regular closed set A of X_1, when $f^{-1}(B) \leqslant A$, there exists a fuzzy semiclosed set C of X_2, such that $B \leqslant C$ and $f^{-1}(C) \leqslant A$.

References

[1] C.L.Chang, J. Math. Anal. Appl. 24(1968), 182-190.

[2] K.K.Azad, J. Math. Anal. Appl. 82(1981), 14-32.

[3] T.H.Yalvac, J. Math. Anal. Appl. 132(1988), 356-364.

[4] Pu Pao Ming and Liu Ying Ming, J. Math. Anal. Appl. 76(1980), 571-599.

[5] Bai Shi Zhong, Fuzzy weakly semicontinuity, Fuzzy Sets and Systems, in press.

Reference

1. A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. I. Math. Surveys No. 7 (Am. Math. Soc., Providence, RI, 1961).
2. N. Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Paul. 28: 17 — 21 (1979).
3. N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5: 203 — 215 (1981).
4. N. Kuroki, Fuzzy semiprime ideals in semigroups, Fuzzy Sets and Systems 8: 71 — 79 (1982).
5. N. Kuroki, On fuzzy semigroups, Inform. Sci. 53: 203 — 236 (1991).
6. Wang Xue-ping, Mo Zhi-wen and Liu Wang-jin, Fuzzy ideals generated by fuzzy point in semigroups, to appear.
7. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35: 512 — 517 (1971).
8. L. A. Zadeh, Fuzzy sets, Inform. and Control 8: 338 — 353 (1965).
9. P. M. Pu and Y. M. Liu, Fuzzy topology I, J. Math. Anal. Appl. 76: 571 — 599 (1980).