Fuzziness vs. randomness, Falling shadow theory*

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Abstract

Even though fuzzy engineering has rapidly developed in Japan and other countries, we have to contend with probabilitists that fuzzy sets theory has its own independent meaning of development. This paper aims to state the essential difference and the duality relationship between the two kinds of uncertainties by means of the Falling Shadow theory. Based on this theory, we can define, not subjectively make, the fundamental fuzzy operations: Union(joint), Intersection(meet), and implication. The idea of couple of squares will pay an important role in application: different degree relations cause different formulae of fuzzy operations.

1. What's the difference between fuzziness and randomness?

Randomness intends to break the law of causality. For example, throw a coin on a table, which side of it will be occur? To answer this question, we can gather those factors who has influence to the result such as Shape of coin, Action of hand, Character of table,...etc.,. Each factor has its own *states space* which consists of all possible states respect to it. Then the Cartesian product space of their states spaces can be used to represent the *fundamental space* Ω termed in probability theory. Which is the domain of the *hidden causing variable* ω; any a fixed point of it requires a determination of the state respect to all factors. Here we can set a **deterministic hypothesis**: For each point ω in Ω, there is one and only one result corresponded, head or tail in this example. If it is not so, then there are definitely some factors having influences to the result but being lost in

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*Reported on the 1990' Meeting of Chinese Natural Science Foundation United Profect "Fuzzy Information Processing and Fuzzy Computing", Beijing in consideration. Add them into the fundamental space such that the hypothesis is eventually held. By this hypothesis,

the possible results, 'head' or 'tail' in the example, form a clear division of Ω . So called a *condition* is a constrain on the domain of hidden variable ω. It can be considered as S, a subset of Ω. Let the set of ω which causes the result of occurring head is denoted as A, When $S \subseteq A$ or $S \cap A = \emptyset$ (see Fig. 1),the result is unique whenever the variable ω varies within S, and there is no randomness! In this case we say that the condition S is *sufficient* for the results. Sufficient condition causes the results of *causality*. So called an *insufficient condition* S means that $S \cap A \neq \emptyset$ and $S \cap A^c \neq \emptyset$. In this case, ω can belong to either A or A^c, and then we cannot predicate if head or tail will be occurred. This is the *randomness*. Randomness occurs in the process when the condition is not sufficient.

Fuzziness intends to break the law of excluded middle. It is not caused by the lack of causality but of hard division. For example, if the coin mentioned above is too old to distinguish between its two sides, then even though the coin we faced has been thrown down yet, we also don't know which side of it is occurred. This is a new kind of uncertainty. Any concept comes from comparing and dividing. Like the division of men and women forms clearly the concepts MAN and WOMAN, a hard division makes clear boundary between the extensions of a concept and its opposite. Unfortunately, most of conceptual divisions are not hard but soft. For example, the division between HEALTHY and NOT HEALTHY is soft. Somebody who is not completely healthy and is not completely not healthy is in a middle state between the two poles. This situation breaks the law of excluded middle; this is the *fuzziness*. It occurs not in the process of predication but of recognition.

Probability theory aims to deal with randomness phenomenon, it grasps the generalized law of causality from the lack of causality; Fuzzy sets theory aims to deal with fuzziness phenomenon, it grasps the generalized law of excluded middle from the middle states.

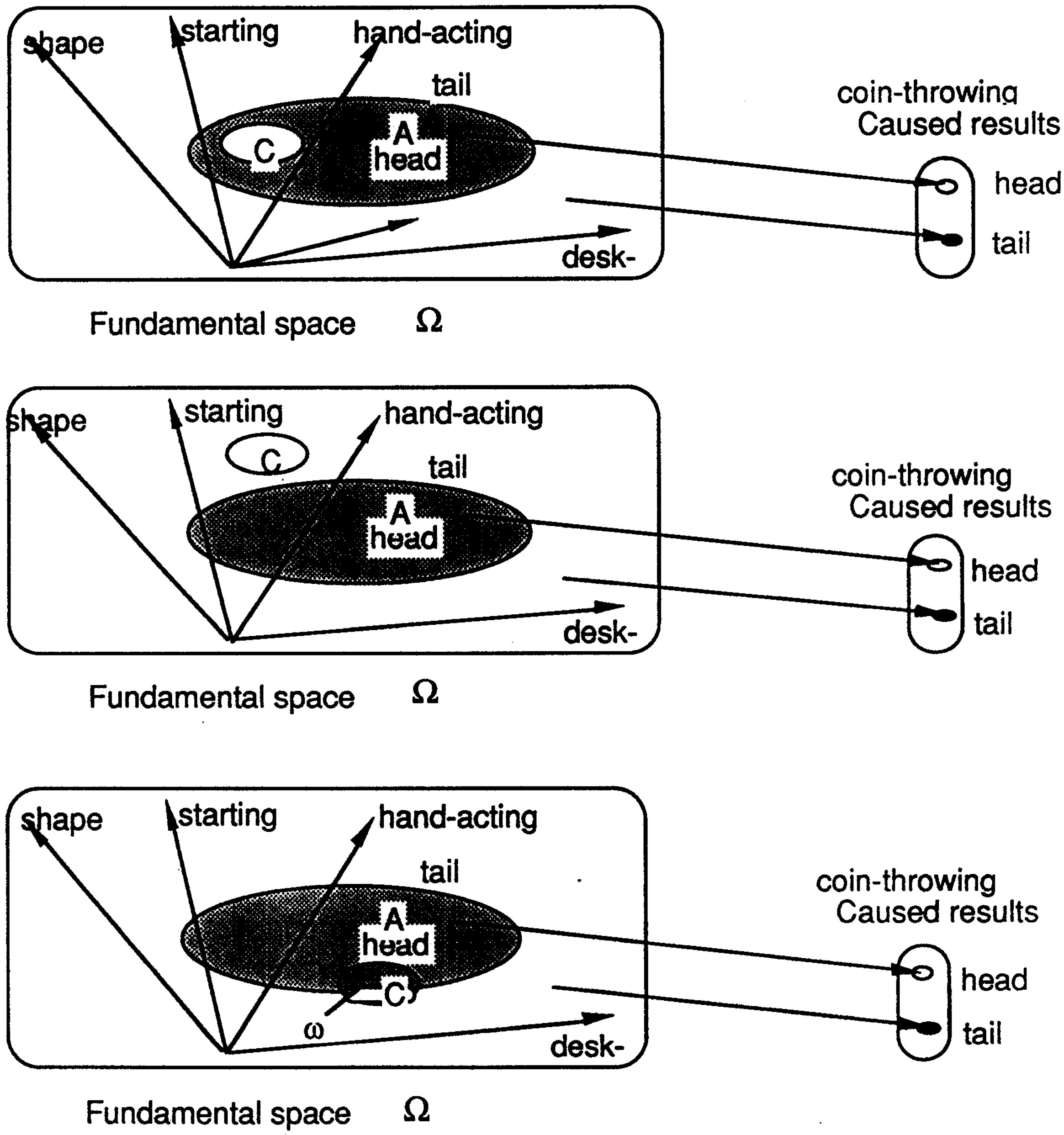

Fig. 1

2. What's the relationship between fuzziness and randomness?

Even though there is essential difference between randomness and fuzziness, there is closed relationship between them still. In the early eighties, Goodman[1][2] and the author of this paper[3] pointed out this relationship independently: A fuzzy subset can be viewed as the *falling shadow*, or *covering function* of a class of random sets. The author getting the idea was not stimulated by a pure mathematical wisdom but by the practical effect of Zhang Nan-lun's *Fuzzy Statistics experiments*[13]

Zhang treated a fuzzy subset as an ordinary subset with a movable boundary. For example, the extension of the fuzzy concept YOUNG in the age-universe can be viewed as a movable interval on the age-axis. He made statistics for the movable interval and accounted its covering frequency for a fixed age, then used it to estimate the membership degree of this age respect to YOUNG. He had successfully taken a lot of experiments and found out the law of stability of covering frequency. The author thought that the intervals in his statistics are the realization of random sets, and promoted his experiments into theoretical analysis. Then, presented the *set-valued statistics and falling shadow theory.*[4][6][7]

The relationship between randomness and fuzziness is a kind of duality. Roughly speaking, a probabilitic statistic experiment is made of '*a fixed circle and a moved point*', where the circle stands for a event A, and point stands for a hidden variable ω; While a fuzzy statistics experiment is made of '*a fixed point and a movable circle*', where the point stands for an object and the circle stands for a concept. (see Fig. 2)

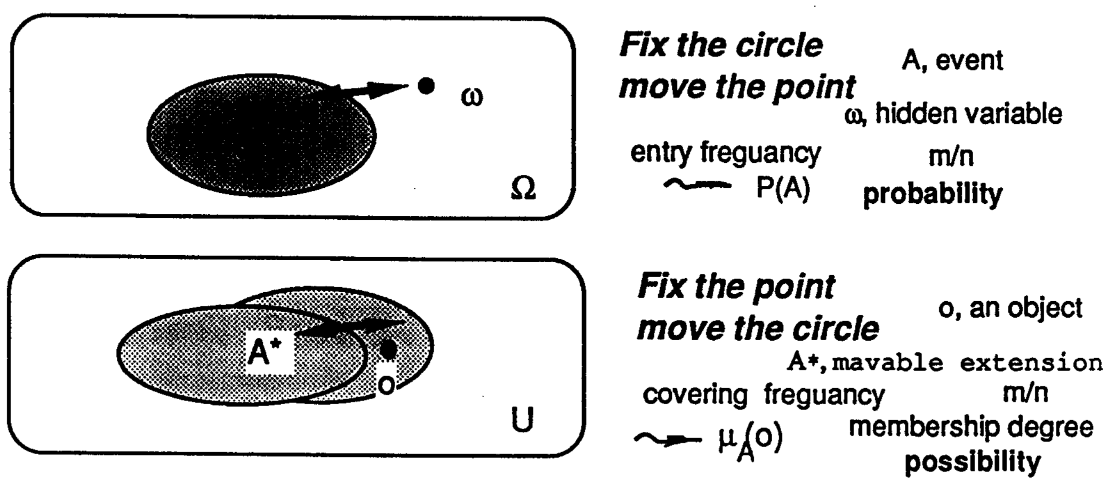

Fig. 2

3. Falling shadow theory

Suppose that $(U,\mathcal{B})$ is a measurable space, and $(\Omega,\mathcal{F},p)$ is a probability field. For any element u in U, denote that

$$\underline{u} = \{ A \in \mathcal{B} \mid u \in A \in \mathcal{B} \}$$

and

$$\underline{C} = \{ \underline{u} \mid u \in C \}$$

$\mathcal{B}$ is called a *hyper σ-field* on U if it is an σ-field on $\mathcal{P}(U)$, and it contains $\underline{U}$. A mapping $\xi:\Omega \text{---}> \mathcal{P}(U)$ is called a *random set* on U if it is $\mathcal{F}$-$\mathcal{B}$ measurable. Because of that $\mathcal{B} \supseteq \underline{U}$, so that for any $u \in U$, the event

$$\mathcal{A} = \{\omega \mid \xi(\omega) \in \underline{u}\} = \{\omega \mid \xi(\omega) \ni u\}$$

is a measurable event and has a determinate probability, it forms a mapping from U to [0,1], the *covering function* of ξ which can be viewed as the membership function of fuzzy subset A_ξ. The later is called the *falling shadow* of ξ:

$$\mu_{A_\xi}(u) = P\{\omega \mid \xi(\omega) \ni u\}.$$

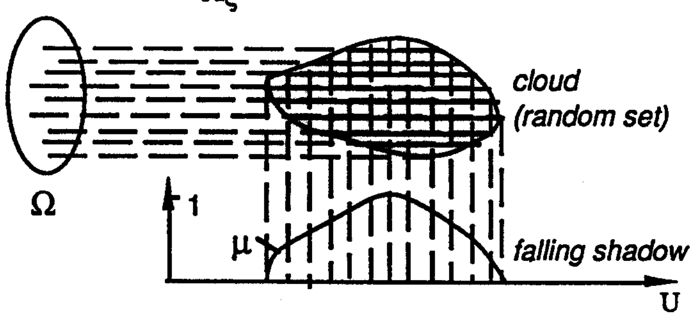

Fig. 3

Random sets can be viewed as clouds, Like the sun shins vertically down, the more thick the cloud, the more high the darkness of the shadow. Given a fuzzy subset A on U, there are infinite clouds which take A as its falling shadow. The most natural selection is the *cut-cloud* which has been presented by Goodman in first and was renamed by us later. Each random set has its fundamental space Ω. When it transfers the probability distribution from $\mathcal{F}$ to $\mathcal{B}$, the probability distribution can be always converted onto the degree interval [0,1]. So that we can chose $([0,1], \mathcal{B}^0, m)$ as the natural probability field, where $\mathcal{B}^0$ be the Borel field on [0,1]. Suppose that the membership function of A is $\mathcal{F}$-measurable, then define

$$\xi(s) = A_s, \qquad (s \in [0,1])$$

while A_s be the cut set of A:

$$A_s = \{ u \in U \mid \mu_A(u) \le s \}$$

It is not difficult to prove that ξ is a random set on [0,1]. This random set is called the *cut-cloud* of A

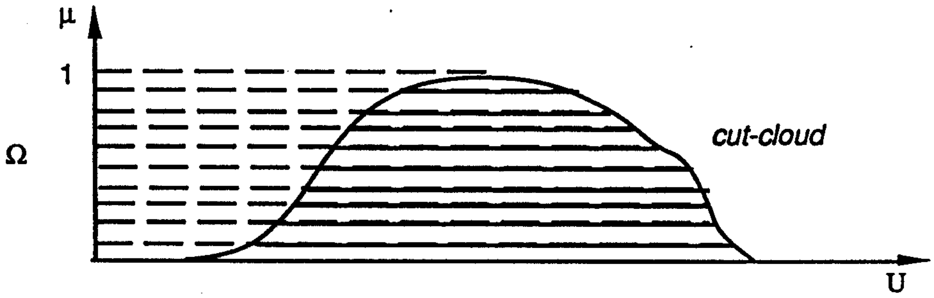

Fig.4

4. Definitions Of fuzzy operations

Let A and B be two fuzzy subsets on the universe U. Let ξ_i(i=1,2) be the cut clouds of them respectively. Because of the variety of relationships between ξ_1 and ξ_2, it is best that they are laid on different probability fields (X_i, $\mathcal{B}^0{}_i, m_i$) (i=1,2). Where $X_i = [0,1]$, $\mathcal{B}^0{}_i = \mathcal{B}^0$ and $m_i = m$ for i=1,2 .

As viewed in Fig.5, there is a product measurable space ($[0,1]^2$, $\mathcal{B}^{o2}$). Suppose that the joint probability field is ($[0,1]^2$, $\mathcal{B}^{o2}$, p), we can redefine ξ_i (i=1,2) on the united probability field as follows:

$$\xi_1': [0,1]^2 \dashrightarrow \mathcal{P}(U)$$
$$(s,t) \longmapsto s \longmapsto \xi_1(s) = A_s$$

$$\xi_2': [0,1]^2 \dashrightarrow \mathcal{P}(U)$$
$$(s,t) \longmapsto t \longmapsto \xi_2(s) = B_t$$

In brief, we still denote ξ_i' as ξ_i (i=1,2).

we can find out the images of ξ_1 and ξ_2 on the diagonal of U^2:

$$\xi_1(s,t) = A_s = (a,b) \quad \xi_2(s,t) = B_t = (c,d)$$

They are two ordinary subsets, then we can take ordinary set-operation to get their union and intersection on the diagonal.

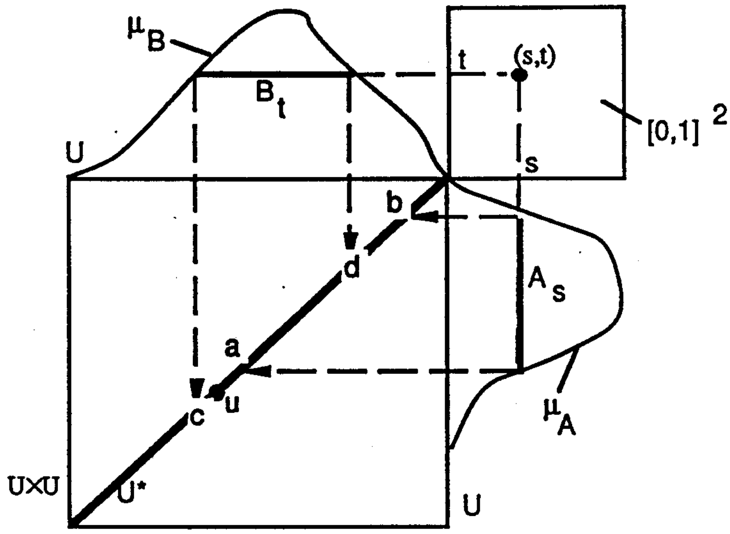

Fig.5

DEFINITION 1 The union and intersection of random sets ξ_1 and ξ_2, denoted as $\xi_1 \cup \xi_2$,$\xi_1 \cap \xi_2$ respectively, is defined as follows:

$$[\xi_1 \cup \xi_2](s,t) = \xi_1(s,t) \cup \xi_2(s,t) = A_s \cup B_t$$

$$[\xi_1 \cap \xi_2](s,t) = \xi_1(s,t) \cap \xi_2(s,t) = A_s \cap B_t$$

We have promoted fuzzy subsets A,B from U, the '*underground*' to $\mathcal{P}$(U), the '*sky*', and we have made union, intersection operations in the sky. If we consider their falling shadows on U, then we can get the natural way of defining union and intersection of fuzzy subsets.

DEFINITION 2 The union and intersection of A and B are defined as the falling shadows of random sets $\xi_1 \cup \xi_2$ and $\xi_1 \cap \xi_2$ respectively. i. e.,

$$\mu_{A\cup B}(u) = p((s,t) \mid \xi_1 \cup \xi_2(s,t) \ni u) = p((s,t) \mid u \in A_s \cup B_t)$$

$$\mu_{A\cap B}(u) = p((s,t) \mid \xi_1 \cap \xi_2(s,t) \ni u) = p((s,t) \mid u \in A_s \cap B_t)$$

Similarly, instead of discussing in a same universe U, When two different universes U and V are concerned with, we can define the operation of implication. The couple of squares has to make a little change: the square U×U is changed into U×V. (see Fig.6)

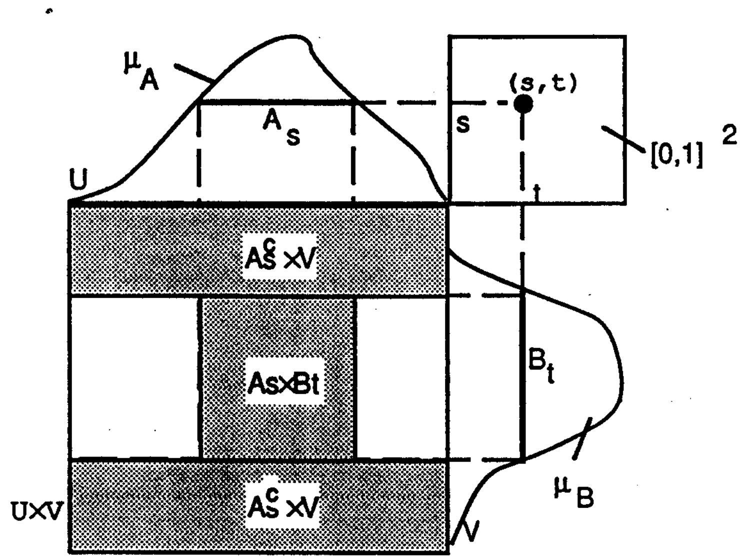

Fig.6

DEFINITION 1' The implication of random sets ξ_1 and ξ_2, denoted as $\xi_1 \rightarrow \xi_2$, is defined as follows:

$$[\xi_1 \rightarrow \xi_2](s,t) = \xi_1(s,t) \rightarrow \xi_2(s,t) = A_s \times B_t \cup A_s^c \times V$$

DEFINITION 2' The implication of A and B, denoted as $A \rightarrow B$ is defined as the falling shadows of $\xi_1 \rightarrow \xi_2$

$$\mu_{A \rightarrow B}(u,v) = p((s,t) \mid \xi_1 \rightarrow \xi_2(s,t) \ni (u,v))$$

5. The joint degrees distributions

A joint probability distribution p satisfying the following conditions :

$$p(A \times [0,1]) = m_1(A); \quad p([0,1] \times B) = m_2(B) \quad (A,B \in \mathcal{B}^0)$$

is called *marginal-uniform joint distributions*(MUJD)

There are infinite possibilities to assign such joint distribution. For examples:

1) p is uniformly distributed on the diagonal of $[0,1]^2$, in this case, we call A,B are *completely positive related.*

2) p is uniformly distributed on the anti-diagonal of $[0,1]^2$, in this case, we call A,B are *completely negative related.*

3) p is uniformly distributed on $[0,1]^2$, in this case we call A,B are independent.

It is obviously that the linear combinations of MUJD s are also a MUJD.

If A and B are completely positive related then we can prove that

$$\mu_{A\cup B}(u) = \max(\mu_A(u), \mu_B(u))$$

$$\mu_{A\cap B}(u) = \min(\mu_A(u), \mu_B(u))$$

$$\mu_{A\to B}(u) = \min(1-\mu_A(u)+\mu_B(u), 1)$$

If A and B are completely negative related, then we can prove that

$$\mu_{A\cup B}(u) = \min(\mu_A(u) + \mu_B(u), 1)$$

$$\mu_{A\cap B}(u) = \max(\mu_A(u)+\mu_B(u)-1, 0)$$

$$\mu_{A\to B}(u) = \max(1-\mu_A(u), \mu_B(u))$$

If A and B are independent, then we can prove that

$$\mu_{A\cup B}(u) = \mu_A(u) + \mu_B(u) - \mu_A(u)\,\mu_B(u)$$

$$\mu_{A\cap B}(u) = \mu_A(u)\,\mu_B(u)$$

$$\mu_{A\to B}(u) = 1 - \mu_A(u) + \mu_A(u)\,\mu_B(u)$$

7. Conclusion

Set-operation is closely relaid on the relationship between operated subsets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logic.

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