On the state space of soft fuzzy algebras

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Abstract. Let $\mathcal{F}$ be a soft fuzzy algebra (an s. f. algebra) and let $S(\mathcal{F})$ be the set of all states on $\mathcal{F}$. In this paper we discuss properties of $S(\mathcal{F})$ which seem of importance in potential applications of s. f. algebras in quantum theories or "soft" sciences (see e.g. [2], [3], [7], [8], [9], [10], [11], [12] and [13]). The general line of our investigation is the analysis of the behaviour of the state space in the situation when we pass from an s. f. algebra to its enlargement. In the first paragraph we prove a representation theorem for the state space of an s. f. algebra. Then, in the second part, we describe states on the product of s. f. algebras. In the third part we take up, and then clarify, the question of how to enlarge s. f. algebras to s. f. algebras with given state spaces. Finally, we prove an extension theorem for states on s. f. algebras.

0 Preliminaries (basics on soft fuzzy algebras and states)

Let us first state the definition of the basic notion we shall deal with in this paper (see e.g. [11]).

Definition 0.1 : *Let X be a non-empty set. A* soft fuzzy algebra *on X is a set $\mathcal{F} \subset [0,1]^X$ satisfying the following conditions:*

1. *the constant zero function belongs to $\mathcal{F}$,*
2. *if $a \in \mathcal{F}$, then $a^\perp = 1 - a \in \mathcal{F}$,*
3. *if $a, b \in \mathcal{F}$, then $a \vee b \in \mathcal{F}$ (the symbol $\vee$ means here the pointwise supremum of functions),*
4. *the constant function 1/2 does not belong to $\mathcal{F}$.*

It can be seen easily that an s. f. algebra is a distributive de Morgan lattice with a least element, 0, and a greatest element, 1. The mapping $^\perp$, an *orthocomplementation*, is an order antiisomorphism. The function $a \wedge a^\perp$, for any $a \in \mathcal{F}$, does not exceed 1/2 (obviously, it does not have to be 0 in general).

In what follows, we shall be frequently interested in questions concerning "enlargements" of s. f. algebras. The notion of an enlargement of an s. f. algebra — intuitively plausible — brings the following definition.

Definition 0.2 : *Let $\mathcal{F}$, $\mathcal{G}$ be s. f. algebras. Then $\mathcal{G}$ is said to be an* enlargement *of $\mathcal{F}$ if there exists a mapping $e : \mathcal{F} \to \mathcal{G}$ such that the following conditions are satisfied:*

1. *the mapping* e *is injective and* $e(0) = 0$,

2. *for each* $a \in \mathcal{F}$, $e(a^\perp) = e(a)^\perp$,

3. *for each* $a, b \in \mathcal{F}$, $e(a \vee b) = e(a) \vee e(b)$.

Observe that if $\mathcal{G}$ is an enlargement of $\mathcal{F}$ and if e is the enlargement mapping then $e(\mathcal{F})$ is an s. f. algebra in its own right, as needed. Thus, the s. f. algebra $\mathcal{F}$ may be viewed as a subset of $\mathcal{G}$. In the dual expression of this, we call $\mathcal{F}$ a *soft fuzzy subalgebra* (abbr. an s. f. subalgebra) of $\mathcal{G}$.

Another important notion of this investigation will be the notion of a state.

Definition 0.3 : *A* state *on* $\mathcal{F}$ *is a mapping* $s : \mathcal{F} \to [0,1]$ *such that*

1. *if* $a \in \mathcal{F}$, *then* $s(a \vee a^\perp) = 1$,

2. *if* $a, b \in \mathcal{F}$ *are orthogonal elements (i.e., if* $a \leq b^\perp$*), then* $s(a \vee b) = s(a) + s(b)$.

Thus, a state on $\mathcal{F}$ is an alternative expression for a finitely additive probability measure on $\mathcal{F}$ (this language is standardly used in quantum theories and elsewhere – see e.g. [12]). Let us denote by $S(\mathcal{F})$ the set of all states on $\mathcal{F}$. The following two propositions bring some elementary properties of s. f. algebras. (In the following propositions and in all what follows, let us use the following notation: $W_0(\mathcal{F}) = \{a \wedge a^\perp : a \in \mathcal{F}\}$, $W_1(\mathcal{F}) = \{a \vee a^\perp : a \in \mathcal{F}\}$.

Proposition 0.4 : *Suppose that* $a, b \in \mathcal{F}$. *Then each* $s \in S(\mathcal{F})$ *satisfies the following properties:*

1. *if* $b \in W_0(\mathcal{F})$, *then* $s(a \vee b) = s(a)$,

2. *if* $b \in W_1(\mathcal{F})$, *then* $s(a \wedge b) = s(a)$.

The proof is elementary.

Proposition 0.5 : *Let us view* $S(\mathcal{F})$ *as a subset of the topological product* $[0,1]^{\mathcal{F}}$. *Then* $S(\mathcal{F})$ *forms there a compact convex subset.*

Proof. The state space is obviously closed under the formation of convex combinations. To show that it is compact, one only has to take into account that the considered topology of $S(\mathcal{F})$ is the pointwise one. The proof of compactness then reduces to verifying that $S(\mathcal{F})$ is closed under the formation of pointwise limits, which is straightforward.
□

Let us observe in the conclusion of preliminaries that if $\mathcal{F} \subset \{0,1\}^X$, then $\mathcal{F}$ is a Boolean algebra and a state on $\mathcal{F}$ is an ordinary finitely additive probability measure.

1 A representation theorem for the state space of a soft fuzzy algebra

In what follows, let $\mathcal{F}$ denote an s. f. algebra on a set X. For each $a \in \mathcal{F}$, let us write $H(a) = a^{-1}((1/2,1])$ ("high values"), $L(a) = a^{-1}([0,1/2))$ ("low values") and $M(a) = a^{-1}(\{1/2\})$ ("middle values"). Thus, $X = H(a) \cup L(a) \cup M(a)$ for each $a \in \mathcal{F}$. Let us now define a mapping $\Xi : [0,1]^X \to \{0,1/2,1\}^X$ as follows:

$$\Xi a(x) = \begin{cases} 0 & \text{if} \quad x \in L(a), \\ 1/2 & \text{if} \quad x \in M(a), \\ 1 & \text{if} \quad x \in H(a). \end{cases}$$

Write $\Xi\mathcal{F} = \{\Xi a : a \in \mathcal{F}\}$. Then $\Xi\mathcal{F} \subset \{0,1/2,1\}^X$ and $\Xi\mathcal{F}$ is an s. f. algebra. Let us call every s. f. algebra with the latter properties a ***three-valued s. f. algebra***. Let us now consider the mapping $\Xi : \mathcal{F} \to \Xi\mathcal{F}$. It can be shown easily that Ξ is a homomorphism(i.e., Ξ preserves 0, the suprema and the orthocomplements). Though Ξ does not have to be injective in general, we shall see that it preserves states. We shall need some auxiliary results. (The following two propositions were essentially obtained in [11]. Since our approach will differ from that of [11], let us sketch their proofs. Following [7], let us define a relation $\sim$ on $\mathcal{F}$ such that $a \sim b$ iff both elements $a \wedge b^{\perp}, a^{\perp} \wedge b$ belong to $W_0(\mathcal{F})$. Thus, $a \sim b$ iff $(a \wedge b^{\perp}) \vee (a^{\perp} \wedge b) \in W_0(\mathcal{F})$.)

Proposition 1.1 : *Suppose that $a, b \in \mathcal{F}$ and suppose further that s is a state on $\mathcal{F}$. If $a \sim b$, then $s(a) = s(b)$.*

Proof. According to Prop. 0.4, $s(a \wedge b) = s((a \wedge b) \vee (a \wedge b^{\perp})) = s(a)$. The equality $s(a \wedge b) = s(b)$ derives similarly.
□

Proposition 1.2 : *Let a,b be such elements in $\mathcal{F}$ that $H(a) \wedge H(b) = \emptyset$. Then there are two orthogonal elements $c, d \in \mathcal{F}$ such that $a \sim c$, $b \sim d$ and $a \vee b \sim c \vee d$. Thus, for each state s on $\mathcal{F}$, $s(a \vee b) = s(a) + s(b)$.*

Proof. Let us put $c = a \wedge b^{\perp}$ and $d = a^{\perp} \wedge b$. As $c \leq a$, $d \leq a^{\perp}$, the elements c, d are orthogonal. We have $a^{\perp} \wedge c \in W_0(\mathcal{F})$ and $a \wedge c^{\perp} = a \wedge (a^{\perp} \vee b) = (a \wedge a^{\perp}) \vee (a \wedge b) \in W_0(\mathcal{F})$. So, $a \sim c$. Analogously, $b^{\perp} \wedge d, b \wedge d^{\perp} \in W_0(\mathcal{F})$. So, $b \sim d$. We also obtain $(a \vee b) \wedge (c \vee d)^{\perp} = (a \wedge c^{\perp} \wedge d^{\perp}) \vee (b \wedge c^{\perp} \wedge d^{\perp}) \in W_0(\mathcal{F})$ and $(a \vee b)^{\perp} \wedge (c \vee d) = (a^{\perp} \wedge b^{\perp} \wedge c) \vee (a^{\perp} \wedge b^{\perp} \wedge d) \in W_0(\mathcal{F})$. Thus, $a \vee b \sim c \vee d$. According to Prop. 1.1, $s(a \vee b) = s(c \vee d) = s(c) + s(d) = s(a) + s(b)$.
□

Proposition 1.3 : *If t is a state on $\Xi\mathcal{F}$, then the mapping $t \circ \Xi$ is a state on $\mathcal{F}$. Conversely, each state on $\mathcal{F}$ can be expressed in the latter manner.*

Proof. The first statement follows immediately from the fact that Ξ is a homomorphism. Suppose now that s is a state on $\mathcal{F}$. According to Prop. 1.1, the mapping $t : \Xi\mathcal{F} \to [0,1]$ determined by the formula $t(\Xi a) = s(a)$ is well defined. We have $t(\Xi a \vee (\Xi a)^{\perp}) = t(\Xi(a \vee a^{\perp})) = s(a \vee a^{\perp}) = 1$. It remains to prove the property 2 of Def. 0.3. For this, let $\Xi a, \Xi b$ be orthogonal elements in $\Xi\mathcal{F}$. Then the sets $H(a), H(b)$, are disjoint and Prop. 1.2 yields

$$t(\Xi a) + t(\Xi b) = s(a) + s(b) = s(a \vee b) = t(\Xi(a \vee b)) = t(\Xi a \vee \Xi b).$$

Thus, we have proved that t is a state on $\Xi\mathcal{F}$.
□

Let us denote by $\mathcal{A}(\mathcal{F})$ the Boolean algebra of subsets of X generated by the collection $H(\mathcal{F}) = \{H(a) : a \in \mathcal{F}\}$. For each $a \in \mathcal{F}$, the collection $\mathcal{A}(\mathcal{F})$ contains also the sets $L(a) = H(a^{\perp})$ and $M(a) = X \setminus (H(a) \cup H(a^{\perp}))$. Thus, the functions Ξa $(a \in \mathcal{F})$, are measurable with respect to $\mathcal{A}(\mathcal{F})$ and, as one sees easily, $\mathcal{A}(\mathcal{F})$ is the smallest algebra on which all the functions Ξa are measurable. Put $\Delta(\mathcal{F}) = \{A \in \mathcal{A}(\mathcal{F}) : \text{there exists } a \in \mathcal{F} \text{ such that } A \subset M(a)\}$. Observe that $\Delta(\mathcal{F})$ is an ideal in $\mathcal{A}(\mathcal{F})$ (see e.g. [14] for the notion of an ideal in a Boolean algebra). Indeed, $\emptyset = M(0) \in \Delta(\mathcal{F})$. Further, if $A \in \mathcal{A}(\mathcal{F})$ and $A \subset B \in \Delta(\mathcal{F})$, then $A \in \Delta(\mathcal{F})$. Finally, if $A, B \in \Delta(\mathcal{F})$, then there are elements $a, b \in \mathcal{F}$ such that $A \subset M(a), B \subset M(b)$. The elements $c = a \wedge a^{\perp}$, $d = b \wedge b^{\perp}$ satisfy $M(c) = M(a) \supset A$ and $M(d) = M(b) \supset B$. Therefore, $M(c \vee d) = M(c) \cup M(d) \supset A \cup B$. Thus, $A \cup B \in \Delta(\mathcal{F})$. Let us now denote by $\mathcal{B}(\mathcal{F})$ the factor algebra $\mathcal{A}(\mathcal{F})/\Delta(\mathcal{F})$ and by φ the corresponding factorization mapping.

Proposition 1.4 : *The mapping $\varphi \circ H$ is a homomorphism of $\mathcal{F}$ onto $\mathcal{B}(\mathcal{F})$.*

Proof. Since φ is a homomorphism and since H preserves 0, the ordering and the suprema, it remains to prove that $\varphi \circ H$ preserves the orthocomplements. However, we have

$$\varphi(H(a^{\perp})) = \varphi(L(a)) = \varphi(L(a) \cup M(a)) = \varphi(H(a)^{\perp}) = (\varphi(H(a)))^{\perp}.$$

So, $\varphi \circ H$ is a homomorphism and $\varphi(H(\mathcal{F})) = \{\varphi(H(a)) : a \in \mathcal{F}\}$ is a subalgebra of $\mathcal{B}(\mathcal{F})$. As $H(\mathcal{F})$ generates $\mathcal{A}(\mathcal{F})$, the collection $\varphi(H(\mathcal{F}))$ generates $\mathcal{B}(\mathcal{F})$ and therefore $\varphi(H(\mathcal{F})) = \mathcal{B}(\mathcal{F})$.
□

The main result of this paragraph is the following representation theorem. Though it may formally resemble the representation obtained by Piasecki [11], the algebra $\mathcal{A}(\mathcal{F})$ used in our attitude essentially differs from that of [11] which, in turn, makes possible to obtain a complete characterization.

Theorem 1.5 : *Let m be a finitely additive probability measure on $\mathcal{B}(F)$. Then the mapping $s = m \circ \varphi \circ H$ is a state on $\mathcal{F}$. Conversely, for each state s on $\mathcal{F}$ there is a unique finitely additive probability measure m on $\mathcal{B}(\mathcal{F})$ such that $s = m \circ \varphi \circ H$.*

Proof. The first statement follows immediately from Prop. 1.4. Suppose that s is a state on $\mathcal{F}$. According to Prop. 1.4, each element of $\mathcal{B}(\mathcal{F})$ is of the type $\varphi(H(a))$ $(a \in \mathcal{F})$. Let us define a mapping $m : \mathcal{B}(\mathcal{F}) \to [0,1]$ by the formula $m(\varphi(H(a))) = s(a)$. We first have to prove that m is well defined. Suppose that $\varphi(H(a)) = \varphi(H(b))$ for some elements $a, b \in \mathcal{F}$. Then we can find an $D \in \Delta(\mathcal{F})$ such that $H(a) \setminus D = H(b) \setminus D$ and, moreover, we can also find a $d \in \mathcal{F}$ such that $D \subset M(d)$. This yields $H(a) \setminus M(d) = H(b) \setminus M(d)$. The element $c = d \vee d^{\perp}$ satisfies $M(c) = M(d)$. Also, $c \in W_1(\mathcal{F})$. We therefore obtain $H(a \wedge c) = H(a) \cap H(c) = H(a) \setminus M(c) = H(b) \setminus M(c) = H(b \wedge c)$, which implies $s(a) = s(a \wedge c) = s(b \wedge c) = s(b)$. We have proved that m is well defined.

Let us now prove that m is a probability measure on $\mathcal{B}(\mathcal{F})$. Trivially, $m(1) = m(\varphi(H(1))) = s(1) = 1$. It remains to verify the additivity of m. Let $\varphi(H(a)), \varphi(H(b))$ be orthogonal elements in $\mathcal{B}(\mathcal{F})$. Then there is a $D \in \Delta(\mathcal{F})$ such that $H(a) \cap H(b) \subset D$ and we can find a $d \in \mathcal{F}$ such that $D \subset M(d)$. Also, there is a $c \in W_1(\mathcal{F})$ (namely, $c = d \vee d^{\perp}$) such that $D \subset M(c)$ $(= M(d))$. We have $H(a) \setminus M(c) = H(a) \cap H(c) = H(a \wedge c)$ and similarly for b, so $H(a \wedge c) \cap H(b \wedge c) = \emptyset$. Applying Prop. 1.2 to the latter collection, we obtain

$$m(\varphi(H(a))) + m(\varphi(H(b))) = s(a) + s(b) = s(a \wedge c) + s(b \wedge c) =$$

$$s((a \wedge c) \vee (b \wedge c)) = s((a \vee b) \wedge c) = s(a \vee b) = m(\varphi(H(a \vee b))).$$

Thus, m is a finitely additive probability measure on $\mathcal{B}(\mathcal{F})$. The uniqueness of m follows immediately from Prop. 1.4.

□

For the need of the next paragraph, let us explicitly formulate the following corollary of Th. 1.5 (see also [7]).

Corollary 1.6 : *Let $\mathcal{F}$ be an s. f. algebra. Then $S(\mathcal{F})$ always contains a two-valued state.*

2 States on the direct sum of soft fuzzy algebras

In this paragraph we find a characterization of states on the direct sum of s. f. algebras. We show that the states on the direct sum are exactly convex combinations of the respective "coordinate" states. Prior to the formulation of the characterization, let us recall the notion of the direct sum of s. f. algebras.

Definition 2.1 : *Let n be a natural number. For any i $(i \leq n)$, let $\mathcal{F}_i$ be an s. f. algebra on a set X_i. Put $X = \bigcup_{i=1}^{n} X_i$ and define the set $\mathcal{F}$ of all fuzzy subsets a of X such that $a \mid X_i \in \mathcal{F}_i$ for all i $(i \leq n)$. Then $\mathcal{F}$ is an s. f. algebra on X. We shall call it the* direct sum of soft fuzzy algebras *$\mathcal{F}_i$ $(i \leq n)$.*

Theorem 2.2 : *Let n be a natural number. For each i $(i \leq n)$, let $\mathcal{F}_i$ be an s. f. algebra on a set X_i and let s_i be a state on $\mathcal{F}_i$. Let $\mathcal{F}$ be the direct sum of $\mathcal{F}_i$ $(i \leq n)$. Then for any choice of non-negative reals $c_1, \ldots, c_n$ with $\sum_{i=1}^{n} c_i = 1$, the mapping s defined by the formula $s(a) = \sum_{i=1}^{n} c_i s_i(a \mid X_i)$ is a state on $\mathcal{F}$. Moreover, all states on $\mathcal{F}$ can be expressed in the latter form.*

Proof. Let us first check that the mapping s described in Th. 2.2 is a state on $\mathcal{F}$. We have, for all $a \in \mathcal{F}$, $s(a \vee a^{\perp}) = \sum_{i=1}^{n} c_i s_i(a \vee a^{\perp} \mid X_i) = \sum_{i=1}^{n} c_i = 1$. Let now $a, b \in \mathcal{F}$ be orthogonal elements. For each $i = 1, \ldots, n$, the elements $a \mid X_i$, $b \mid X_i \in \mathcal{F}_i$ are orthogonal. Hence,

$$s(a \vee b) = \sum_{i=1}^{n} c_i s_i((a \vee b) \mid X_i) = \sum_{i=1}^{n} c_i[s_i(a \mid X_i) + s_i(b \mid X_i)] =$$

$$\sum_{i=1}^{n} c_i s_i(a \mid X_i) + \sum_{i=1}^{n} c_i s_i(b \mid X_i) = s(a) + s(b).$$

Thus, we have proved that s is indeed a state on $\mathcal{F}$.

Conversely, suppose that p is a state on $\mathcal{F}$. For each $i = 1, \ldots, n$, the characteristic function χ_i of the set X_i belongs to $\mathcal{F}$. Obviously, each $a \in \mathcal{F}$ can be expressed in the form $a = \bigvee_{i=1}^{n}(a \wedge \chi_i)$. We see that there is a supremum of an orthogonal sequence on the right-hand side of the latter equality . Thus, $s(a) = \sum_{i=1}^{n} s(a \wedge \chi_i)$. Now, if $s(\chi_i) \neq 0$ for an i ($i \leq n$), then the mapping $s_i : \mathcal{F}_i \to [0,1]$ defined by the formula $s_i(a \mid X_i) = s(a \wedge \chi_i)/s(\chi_i)$ ($a \in \mathcal{F}$) is a state on $\mathcal{F}_i$. We then put $c_i = s(\chi_i)$. If $s(\chi_i) = 0$, we put $c_i = 0$ and we take for s_i an arbitrary state on $\mathcal{F}_i$. We then obtain $s(a) = \sum_{i=1}^{n} c_i s_i(a \mid X_i)$ for all $a \in \mathcal{F}$ and this completes the proof of Th. 2.2.

3 State spaces of enlargements of soft fuzzy algebras

In this paragraph we ask the question of whether an s. f. algebra can be enlarged to an s. f. algebra with a preassigned state space. The question will be answered positively (see Th. 3.1). Prior to that, let us recall that if C_1, C_2 are two convex subsets of R^I (I is a set), then we call C_1, C_2 affinely homeomorphic if there is an affine isomorphism of C_1 onto C_2 which is a topological homeomorphism (see e.g. [1]).

Theorem 3.1 : *Let $\mathcal{F}$ be an s. f. algebra and let $\mathcal{B}$ be a Boolean algebra. Then there is an enlargement $\mathcal{G}$ of $\mathcal{F}$ such that $S(\mathcal{G})$ and $S(\mathcal{B})$ are affinely homeomorphic. A corollary: $\mathcal{F}$ admits enlargements with arbitrary state spaces.*

Proof. We may (and shall) suppose that $\mathcal{F}$ is an s. f. algebra on a set X and $\mathcal{B}$ is a Boolean algebra of subsets of a set Y. Let us denote by $\mathcal{F}_{1/2}$ the least set of functions on X containing $\mathcal{F} \cup \{1/2\}$ and being closed under the formation of the suprema and the orthocomplements. Let us take for $\mathcal{G}$ the set of all functions $g : X \cup Y \to [0,1]$ such that $g \mid X \in \mathcal{F}_{1/2}$ and $g \mid Y$ is a characteristic function of a (crisp) set from $\mathcal{B}$. It is easy to verify that $\mathcal{G}$ is an s. f. algebra. (In fact, it is the direct sum of $\mathcal{F}_{1/2}$ and $\mathcal{B}$.)

In order to find a s. f. subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{F}$, let us first take a two-valued state s on $\mathcal{F}$ (such a state does exist – Cor. 1.6). Let now $i : \mathcal{F} \to \mathcal{G}$ denote the mapping determined by the following two requirements:

1. for each $a \in \mathcal{F}$, $i(a) \mid X = a$,
2. for each $a \in \mathcal{F}$, $i(a) \mid Y = 0$ if $s(a) = 0$, $i(a) \mid Y = 1$ otherwise.

It can be verified easily that i preserves 0, the orthocomplements and the suprema. As i is obviously injective, it has to be an isomorphism of $\mathcal{F}$ onto the s. f. subalgebra $i(\mathcal{F})$ of $\mathcal{G}$.

We have $\Delta(\mathcal{F}) = \{A \in \mathcal{A}(\mathcal{F}) : A \subset X\}$. Hence, the Boolean algebra $\mathcal{B}(\mathcal{F})$ ($= \mathcal{A}(\mathcal{F})/\Delta(\mathcal{F})$) is isomorphic to $\mathcal{B}$. According to Th. 1.5, the state space of $\mathcal{G}$ is affinely homeomorphic to $S(\mathcal{B})$. The proof is complete.
□

For the finite-dimensional state spaces we obtain the following corollary of the latter result.

Corollary 3.2 : *Let n be a natural number. Then every s. f. algebra possesses an enlargement $\mathcal{G}$ such that $S(\mathcal{G})$ is an n-dimensional simplex. In particular, every s. f. algebra possesses an enlargement $\mathcal{G}$ such that $S(\mathcal{G})$ is a singleton.*

Remark 3.3 : It should be noted that an analogous question as we have discussed above has been solved for the σ-additive case in [10]. It turns out that the statement of Th. 3.1 does not allow a formal translation in the σ-additive language.

Remark 3.4 : In connection with the results of this paragraph a natural question appears if we can enlarge s. f. algebras to s. f. algebras with preassigned state space (as we have done here) and, at the same time, with a preassigned Boolean algebra of crisp sets. The answer to this question is not known to the authors for the time being.

4 Extensions of states

In this paragraph we ask if any state on $\mathcal{F}$ can be extended over an enlargement, $\mathcal{G}$, of $\mathcal{F}$. As we have seen (Cor. 3.2), the state space $S(\mathcal{G})$ can in general be very small (in the extreme case it can be a singleton). It follows that if we want to make our question meaningful, we have to assume that $S(\mathcal{G})$ is considerably "rich". This property of $S(\mathcal{G})$ is formally described in the following definition.

Definition 4.1 : *An s. f. algebra $\mathcal{G}$ is called* rich *if for any $a \in \mathcal{G} \setminus W_0(\mathcal{G})$ there is a state $s \in S(\mathcal{G})$ such that $s(a) = 1$.*

We now have the following result.

Theorem 4.2 : *Let $\mathcal{F}$, $\mathcal{G}$ be s. f. algebras and let $\mathcal{G}$ be an enlargement of $\mathcal{F}$. Let $\mathcal{G}$ be rich. Then every state $s \in S(\mathcal{F})$ can be extended over $\mathcal{G}$.*

In other words, the latter theorem asserts that if $s \in S(\mathcal{F})$ then there is a $t \in S(\mathcal{G})$ such that $t \mid \mathcal{F} = s$. In order to prove this result, we first have to analyse the notion of a W-partition in an s. f. algebra.

Definition 4.3 : *A* W-partition *in $\mathcal{F}$ is a finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\mathcal{F}$ such that $H(a_i) \cap H(a_j) = \emptyset$ for all $i \neq j$ and such that $\bigvee_{i=1}^{n} a_i \in W_1(\mathcal{F})$.*

We shall need the following auxiliary results on W-partitions.

Lemma 4.4 : *If $a \in \mathcal{F}$ then the pair $\{a, a^\perp\}$ is a W-partition in $\mathcal{F}$. Thus, each element in $\mathcal{F}$ belongs to a W-partition.*

The proof is elementary.

Lemma 4.5 : *Let $\mathcal{F}$ be an s. f. algebra and let $\mathcal{G}$ be its rich enlargement. Let $\{a_1, a_2, \ldots, a_n\}$ be a W-partition in $\mathcal{F}$ and let s be a state on $\mathcal{F}$. Then there is a state t on $\mathcal{G}$ such that $t(a_i) = s(a_i)$ for all i $(i \leq n)$.*

Proof. Observe first that $W_0(\mathcal{F}) = \mathcal{F} \cap W_0(\mathcal{G})$. Let us now construct the required state t of $\mathcal{G}$. Let us assign, to any a_i of the W-partition, a state $t_i \in S(\mathcal{G})$ in the following manner. If $a_i \notin W_0(\mathcal{F})$, then we take for t_i an arbitrary state of $S(\mathcal{G})$ with $t_i(a_i) = 1$ (the existence of such t_i is ensured by the richness of $\mathcal{G}$). If $a_i \in W_0(\mathcal{F})$, then we take for t_i an arbitrary state of $S(\mathcal{G})$. We then put $t = \sum_{i=1}^{n} s(a_i)t_i$.
□

Lemma 4.6 : *Suppose that $\mathcal{P} = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{Q} = \{b_1, b_2, \ldots, b_k\}$ are two W-partitions in $\mathcal{F}$. Then the collection $\mathcal{R} = \{a_i \wedge b_j : i \leq n,\ j \leq k\}$ is also a W-partition in $\mathcal{F}$. Moreover, if $s, t \in S(\mathcal{F})$ and if both s, t coincide on all the elements of $\mathcal{R}$, then they coincide on all elements of both $\mathcal{P}$ and $\mathcal{Q}$.*

The proof is straightforward.

Let us now return to the proof of Th. 4.2. Suppose that we are given a state $s \in S(\mathcal{F})$. We have to find the extension of s, some $t \in S(\mathcal{G})$. Let us denote by $\mathcal{W}$ the set of all W-partitions in $\mathcal{F}$. If $\mathcal{P} \in \mathcal{W}$, let us denote by $C(\mathcal{P})$ the set of all states on $\mathcal{G}$ which coincide with s on all elements of $\mathcal{P}$. According to Lemma 4.5, for any $\mathcal{P} \in \mathcal{W}$ we have $C(\mathcal{P}) \neq \emptyset$. Moreover, for any pair $\mathcal{P}, \mathcal{Q} \in \mathcal{W}$ there is an $\mathcal{R} \in \mathcal{W}$ such that $C(\mathcal{R}) \subset C(\mathcal{P}) \cap C(\mathcal{Q})$ (Lemma 4.6). Thus, the collection $C(\mathcal{P})$ ($\mathcal{P} \in \mathcal{W}$) is directed. As one sees easily, each $C(\mathcal{P})$ is obviously a closed subset of $S(\mathcal{G})$. Due to the compactness of $S(\mathcal{G})$, there is a state $t \in S(\mathcal{G})$ which belongs to $\bigcap_{\mathcal{P} \in \mathcal{W}} C(\mathcal{P})$ (see e.g. [4]). Since $\bigcup \mathcal{W} = \mathcal{F}$ (Lemma 4.4), we infer that the state t coincides with s on the entire $\mathcal{F}$. This completes the proof.

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