Duality of seminormed fuzzy integral and semiconormed fuzzy integral

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Abstract

This paper shows that the seminormed fuzzy integral and the semiconormed fuzzy integral introduced in /1/ are two dual fuzzy integrals. The dual principles are given. Some groups of dual propositions are listed.

1 Introduction

Suarez and Gil/1/ introduced the concepts of seminormed fuzzy integral and semiconormed fuzzy integral, which are all the extensions of Sugeno's one defined in /3/. A series of propositions to the two fuzzy integrals were proved and some applications were studied in /1, 2/. In this paper, we show that the two fuzzy integrals have the property of duality. The dual principles are given and some groups of dual propositions are listed. In the view of the duality of the two fuzzy integrals, all properties about seminormed(resp. semiconormed) fuzzy integral can be obtained via the properties about semiconormed (resp. semonormed) fuzzy integral.

2 Relative definitions

Definition2.1(/4/) A function $T: [0, 1] \times [0, 1] \to [0, 1]$ is called a t-seminorm(resp. semiconorm), if T satisfies

(1) $T(x, 1) = T(1, x) = x$(resp. $T(x, 0) = T(0, x) = x$), $\forall x \in X$;

(2) $\forall x, y, z, u \in [0, 1]$, $x \leq z$, $y \leq u \Rightarrow T(x, y) \leq T(z, u)$.

Let T and $\perp$ be t-seminorm and t-semiconorm respectively, If $\forall x, y \in [0, 1]$, there holds $T(x, y) + \perp(1-x, 1-y) = 1$, we call T and $\perp$ dual.

Definition2.2(/3/) Let X be a nonempty set, β a σ-algebra of subsets of X. A set function $g: \beta \to [0, 1]$ is called a fuzzy measure, if g satisfies

(1) $g(\emptyset) = 0$, $g(X) = 1$;

(2) $A \subset B \Rightarrow g(A) \leq g(B)$;

(3) $A_n \uparrow$ (or $A_n \downarrow$) $\Rightarrow g(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} g(A_n)$

(X, β, g) is called a fuzzy measure space.

Definition2.3(/5/) Let g_1 and g_2 be two fuzzy measure on (X, β). g_1, g_2 are called dual, if $\forall A \in \beta$, there holds

$$g_1(A) + g_2(\bar{A}) = 1$$

where $\bar{A} = X - A$.

Definition2.4(/1, 2/) Let h be a β-measurable function, $0 \leq h \leq 1$, T a t-seminorm and $\perp$ a semiconorm. We define the seminormed fuzzy integral of h over $A \in \beta$ as

$$\int_A hTg = \sup_{\alpha \in (0,1]} T(\alpha, g(A \cap (h \geq \alpha)))$$

and the semiconormed fuzzy integral of h over $A \in \beta$ as

$$\int_A h\ g = \inf_{\alpha \in [0,1)} \perp(\alpha, g(A \cap (h > \alpha)))$$

Suarez and Gil /1, 2/ shows that $\forall A \in \beta$,

$$\int_A hTg = \int_X (h \wedge I_A) Tg$$

and

$$\int_A h \perp g = \int_X (h \wedge I_A) \perp g$$

where I_A is the characteristic function of A. The above two conclusions tell us that the two kinds of fuzzy integrals over $A \in \beta$ can be changed into the ones over X, of course, the function h must be replaced by $h \wedge I_A$. In the following, we always assume that all fuzzy integrals are considered over X, and for short, we denote $\int_X hTg$ and $\int_X h \perp g$ by $\int hTg$ and $\int h \perp g$ respectively.

3 Duality of seminormed fuzzy integral and semiconormed fuzzy integral

The following Theorem is the principal result of this paper.

Theorem3.1 For every β-measurable function h, every dual t-seminorm T and t-semiconorm $\perp$, every dual fuzzy measures g and $\bar{g}$, there holds

$$\int hTg + \int \bar{h} \perp \bar{g} = 1$$

where $\bar{h}$ is defined as $\bar{h}(x) = 1 - h(x)$, $x \in X$, we call $\bar{h}$ the dual function of h.

Proof. From Definition2.4, we have

$$\int hTg = \sup_{\alpha \in (0,1]} T(\alpha, g(h \geq \alpha)))$$

$$= 1 - \inf_{\alpha \in (0,1]} (1 - T(\alpha, g(h \geq \alpha)))$$

$$= 1 - \inf_{\alpha \in (0,1]} \perp(1-\alpha, 1-g(h \geq \alpha))$$

$$= 1 - \inf_{\alpha \in (0,1]} \perp(1-\alpha, \bar{g}(h < \alpha))$$

$$= 1 - \inf_{\alpha \in (0,1]} \perp(1-\alpha, \bar{g}(\bar{h} > 1-\alpha))$$

$$= 1 - \inf_{1-\alpha\in[0,1)} \perp(1-\alpha,\ \bar{g}(\bar{h}>1-\alpha))$$

$$= 1 - \inf_{\alpha\in[0,1)} \perp(\alpha,\ \bar{g}(\bar{h}>\alpha))$$

$$= 1 - \int \bar{h}\perp\bar{g}$$

and the conclusion follos.

Theorem3.1 describes the relations between two kinds of fuzzy integrals. It is easy to know that if a proposition with respect to one kind of fuzzy integral is proved, we can obtain the relative "dual" proposition with respect to the other, according to the following dual principles :

(1) t-seminorm T $\rightarrow$ the dual t-semiconorm $\perp$;

(2) t-semiconorm $\perp \rightarrow$ the dual t-seminorm T;

(3) fuzzy measure g $\rightarrow$ the dual fuzzy measure $\bar{g}$;

(4) β-measurable function h $\rightarrow$ the dual β-measurable function $\bar{h}$;

(5) the value of the fuzzy integral $\int \rightarrow$ the dual of it: $1 - \int$;

In section 4, we shall list some groups of dual propositions about the two fuzzy integrals.

4. Some groups of dual propositions about two fuzzy integrals

In this section, we use P. to stand for 'Proposition'.

The following groups (1) - (7) are dual propositions about the two fuzzy integrals.

(1) P.4.1 $\quad h_1 \leq h_2 \Rightarrow \int h_1 T g \leq \int h_2 T g$,

P.4.1' $\quad h_1 \leq h_2 \Rightarrow \int h_1 \perp g \leq \int h_2 \perp g$;

(2) P.4.2 $\quad \int aTg = a \quad \forall a \in [0, 1]$,

P.4.2' $\quad \int a\perp g = a \quad \forall a \in [0, 1]$;

(3) P.4.3 $\int (a \vee h)Tg = a \vee \int hTg \quad \forall a \in [0, 1]$,

P.4.3' $\int (a \wedge h)\perp g = a \wedge \int h\perp g \quad \forall a \in [0, 1]$;

(4) P.4.4 $\int I_A Tg = g(A) \quad \forall A \in \beta$,

P.4.4' $\int I_A \perp g = g(A) \quad \forall A \in \beta$;

(5) P.4.5 $\int hT_{21}g \leq \int hTg \leq \int hT_1 g$,

P.4.5' $\int h\perp_{20}g \geq \int h\perp g \geq \int h\perp_1 g$;

(6) P.4.6 Let g be a possibility measure, then

$\int (h_1 \vee h_2)Tg = \int h_1 Tg \vee \int h_2 Tg$

P.4.6' Let g be a consonant belief function, then

$\int (h_1 \wedge h_2)\perp g = \int h_1 \perp g \wedge \int h_2 \perp g$

(7) P.4.7 Let T be a continuous t-seminorm and $h_n \uparrow h$ (resp. $h_n \downarrow h$), then $\int h_n Tg \uparrow \int hTg$ (resp. $\int h_n Tg \downarrow \int hTg$).

P.4.7' Let $\perp$ be a continuous t-semiconorm and $h_n \downarrow h$ (resp. $h_n \uparrow h$), then $\int h_n \perp g \downarrow \int h\perp g$ (resp. $\int h_n \perp g \uparrow \int h\perp g$).

Now, as an example, we prove P.4.7' via P.4.7 by using dual principles.

Suppose P.4.7 is proved, we verify that P.4.7' is true.

We only cosider the case that $h_n \downarrow h$, the other is similar.

Let $h_n \downarrow h$, then $\bar{h}_n \uparrow \bar{h}$. By using P.4.7, we have,

$$\int \bar{h}_n T\bar{g} \uparrow \int \bar{h} T\bar{g}$$

Applying dual principles to above result, we obtain

$$1 - \int h_n \perp g \uparrow 1 - \int h\perp g$$

That is to say

$$\int h_n \perp g \downarrow \int h\perp g$$

References

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