SOME NEW PROPERTIES OF SUGENO'S INTEGRAL.

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This paper deals with some new properties of the fuzzy integral with respect to fuzzy and probability measures.

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1. Introduction.

A concept of a fuzzy measure was defined by Sugeno [1]. This approach generalized probability measure by dropping the addity property and replacing a weaker one, i.e. monotonicity. By using fuzzy measure Sugeno presented his fuzzy integral and fundamental properties is investigated.

Let Ω is an arbitrary set and $\mathcal{B}$ is a Borel field of Ω.

Definition. A set function μ defined on $\mathcal{B}$ that has the following properties is called a fuzzy measure:

1. $\mu(\emptyset)=0, \quad \mu(\Omega)=1$;

2. If $A, B \in \mathcal{B}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$;

3. If $F_n \in \mathcal{B}$ and $\{F_n\}$ is monotone, then $\lim\limits_{n\to\infty} \mu(F_n)=\mu(\lim\limits_{n\to\infty} F_n)$.

Let $X: \Omega \to [0,1]$ is a $\mathcal{B}$- measurable function.

Definition. A fuzzy integral of X over Ω with respect to μ is defined and denoted as following:

$$E_\mu(X)=\int X(\omega)\circ\mu = \sup_{\alpha\in[0,1]} (\min(\alpha, \mu(F_\alpha))),$$

where

$$F_\alpha = \{\omega \in \Omega : X(\omega) > \alpha\}.$$

Fuzzy integral are also called fuzzy expectation.

Let us begin by recalling some properties of the fuzzy integral [1]:

$$E_\mu(\min(X_1,X_2)) \leq \min(E_\mu(X_1), E_\mu(X_2)),$$

$$E_\mu(\max(X_1,X_2)) \geq \max(E_\mu(X_1), E_\mu(X_2)).$$

Instead min,max operations, which are simple, but important triangular norm and conorm, in the next sections will be considered other cases of triangular norms [2].

New [1,3]: $E_{\mu_\lambda}(n_\lambda(X)) = n_\lambda(E_{\mu_\lambda}(X))$,

where μ_λ is Sugeno's λ- fuzzy measure and $n_\lambda(x) = \frac{1-x}{1+\lambda x}$ is Sugeno's λ- negation.

Next connection between the fuzzy expectatin $E_P(X)$ and Lebesque integral with respect to probability measure i.e. classical expection $M_P(X) = \int X\,dP$ was stated [1]: $|E_P(X) - M_P(X)| \leq \frac{1}{4}$.

This comparison make sense, since any probability measure is, in particular, a fuzzy measure.

2. Inequalities for the fuzzy integral.

Our main problem here is to get a Jensen-like inequality.

Theorem 1. If function $g: [0,1] \to [0,1]$ is a single - place continuosly increasing and $g(x) \leq x$, then

$$g(E_\mu(X)) \leq E_\mu(g(X)).$$

Theorem 2. If function $f: [0,1] \to [0,1]$ is a single - place continuosly decreasing and $f(x) \geq x$, then

$$f(E_\mu(X)) \geq E_\mu(f(X)).$$

We'll get from it more interesting corollaries.

Corollary 1. If $1 \leq K \leq m$, then

$$(E_{\mu}(X^{K}))^{\frac{1}{K}} \leq (E_{\mu}(X^{m}))^{\frac{1}{m}}.$$

Corollary 2. Let X_1, X_2 are measurable function and $K \geq 1$, then

$$E_{\mu}(\min(X_1,X_2)) \leq \min(E_{\mu}(X_1^{K}), E_{\mu}(X_2^{K}))^{\frac{1}{K}}.$$

3. Comparison between the fuzzy expectation and the clssical expectation.

We suppose in this section that $\mu = P$ is a probability measure and X is measurable function, then the fuzzy expectation $E_P(X)$ and classical expectation $M_P(X)$ make sence.

Theorem 3. Let X_1, X_2 are measurable functions and $K \geq 1$, then

$$(E_P(\min(1, X_1+X_2)^{K}))^{\frac{1}{K}} \leq \min(1, (M_P(X_1^{K}))^{\frac{1}{2K}} + (M_P(X_2^{K}))^{\frac{1}{2K}}).$$

Theorem 4. Let X_1, X_2 are measurable functions and $K \geq 1$, $m \geq 1$, $\frac{1}{K} + \frac{1}{m} = 1$, then

$$E_P(X_1 X_2) \leq (M_P(X_1^{K}))^{\frac{1}{2K}} (M_P(X_2^{m}))^{\frac{1}{2m}}.$$

Theorem 5. $n(E_P(X)) = E_P(n(X))$, $n(x) = 1-x$.

Note that by using theorems 3,5 we can obtained Sugeno's inequality

$$| E_P(X) - M_P(X) | \leq \frac{1}{4}.$$

Corollary 3. For Yager's operation

$$y(X_1^{K}, X_2^{K}) = \min(1, X_1^{K} + X_2^{K})^{\frac{1}{K}}, \quad K \geq 1,$$

we have

$$E_P(y(X_1^{K}, X_2^{K})) \leq y^{\frac{1}{2}}(M_P^{K}(X_1), M_P^{K}(X_2)).$$

Corollary 4. Let $M_p(X_1X_2) = M_p(X_1)M_p(X_2)$, then for operation

$$R_\lambda(X_1^K, X_2^K) = \min(1, X_1^K + X_2^K + \lambda X_1^K X_2^K)^{\frac{1}{K}}, \quad K \geq 1, \ -1 < \lambda < \infty,$$

we can conclude that the following inequality hold true

$$E_p(R_\lambda(X_1^K, X_2^K)) \leq R_\lambda^{\frac{1}{2}}(M_p^K(X_1), M_p^K(X_2)).$$

Corollary 5. If $\alpha + \beta = 1$, $\alpha > 0$, then

$$E_p(X_1^\alpha X_2^\beta) \leq (M_p(X_1))^{\frac{\alpha}{2}}(M_p(X_2))^{\frac{\beta}{2}}.$$

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