Fuzzy Measure Defined by Pan-Integral

WANG Xizhao and HA Minghu
Mathematics Department, Hebei University, Baoding, Hebei, P.R.CHINA

Massimo in [1] discussed a type of fuzzy measure, its structural properties, corresponding fuzzy integral and convergence theorems and all of results in [1] had been generalized by H. Minghu and W.Xizhao in [2]. In this paper, we introduce a type of fuzzy measure defined by pan-integral and discuss its some structural properties.

Keywords. Fuzzy measure, pan-integral.

Throughout this paper, we make the following conventions: $R^+=[0,\infty]$, $\vee$ denotes the maximum operation on R^+, $(X,F(X),P)$ is a fixed fuzzy measure space and M^+ denotes all of nonegative measurable functions on $(X,F(X))$.

Definition 1.

Let $*$ be a binary operation defined on R^+ with the following properties:

(1.1) $a*b=b*a$ (1.2) $(a*b)*c=a*(b*c)$

(1.3) $(a\vee b)*c=(a*c)\vee(b*c)$ (1.4) $a_i\leqslant b_i,\ a_j\leqslant b_j \Rightarrow a_i*a_j\leqslant b_i*b_j$

(1.5) $a*0=0$ (1.6) $a*b=0 \Rightarrow a=0$ or $b=0$

(1.7) there exists an unit element I in R^+ such that $I*a=a*I=I$

(1.8) $\mathrm{Lim}(a_n*b_n)=\mathrm{Lim}a_n*\mathrm{Lim}b_n$ when $\mathrm{Lim}a_n$ and $\mathrm{Lim}b_n$ exist and are finite for all a, b, c, a_n, b_n $(n=1,2,\ldots\ldots)$ which are in R^+.

Then, $(R^+,\vee,*)$ is called a EP-semiring.

Example. $(R^+,\vee,\wedge)$ and $(R^+,\vee,\cdot)$ are all EP-semirings, their unit elements are $I=+\infty$ and $I=1$ respectively. Where $\wedge$ and $\cdot$ are the minimum operation and multiplication operation on R^+.

Let $(R^+,\vee,*)$ be a EP-semiring. The sextuple system $(R^+,\vee,*,X,F(X),P)$ is called a pan-space.

Definition 2.

The fuzzy measure P is called fuzzy additive if $P(E\cup F)=P(E)\vee P(F)$ for all $E, F\in F(X)$. P is called null additive if $P(E\cup F)=P(E)$ for all $E, F\in F(X)$, $E\cap F=\phi$ and $P(F)=0$. P is called above autocontinuous if $P(E\cup F_n)\rightarrow P(E)$ for all $E,F_n\in F(X)$ $(n=1,2,\ldots)$, $E\cap F_n=\phi$ and $P(F_n)\rightarrow 0$ $(n\rightarrow\infty)$.

Proposition 1.

If P is fuzzy additive ,then it is null additive and above autocontinuous.

Definition 3.

Let $(R^+,\vee,*,X,F(X),P)$ be a pan-space, $f\in M^+$ and $E\in F(X)$. The pan-integral of the function f with respect to P on E is defined by

$$\int_E f dP = \bigvee_{r\in[0,\infty]} (r*P(\{x: f(x)\geq r\}\cap E))$$

Proposition 2.

Let $f,g\in M^+$, $E,F\in F(X)$. Then

(1) $f\leq g \Rightarrow \int_E f dP\leq \int_E g dP$ (2) $E\subset F \Rightarrow \int_E f dP\leq \int_F f dP$

(3) $P(E)=0 \Rightarrow \int_E f dP=0$ (4) $\int_E c dP \geq c*P(E)$.

Where c is a nonegative real number.

Proof. The results can be obtained immediately by definition 3.

Theorem 1.

Let $(R^+,\vee,*,X,F(X),P)$ be a pan-space, $P(X)<\infty$, $f\in M^+$. Define

$$P_f(E)= \int_E f dP \qquad E\in F(X)$$

Then P_f is also a fuzzy measure on $(X,F(X))$.

Proof. Let $P_r(E)=r*P(\{f\geq r\})\cap E)$ for every $E\in F(X)$ and $r\geq 0$. Then

(1) $P_r(\phi)=r*0=0$

(2) $A\subset B \Rightarrow P(\{f\geq r\}\cap A) \leq P(\{f\geq r\}\cap B)$

$\Rightarrow r*P(\{f\geq r\}\cap A) \leq r*P(\{f\geq r\}\cap B)$

$\Rightarrow P_r(A) \leq P_r(B)$.

(3) $A_n\in F(X)$ and $A_n\subset A_{n+1}$ $(n=1,2,\ldots\ldots)$ $=$

$$\Rightarrow P_r(\mathrm{Lim}A_n) = P_r(\bigcup_{n=1}^{\infty} A_n) = r*P(\bigcup_{n=1}^{\infty} (\{f\geq r\}\cap A_n))$$

$$=r*\mathrm{Lim}P(\{f\geq r\}\cap A_n) = \mathrm{Lim}(r*P(\{f\geq r\}\cap A_n)) = \mathrm{Lim}P_r(A_n)$$

As $P_f=\bigvee_r P_r$, it is easy to see that $P_f(\phi)=0$ and $A\subset B$ implies $P_f(A)\leq P_f(B)$. To complete the proof, we need proving the continuity from blow and from above of the set function P_f.

In fact, if $A_n\subset A_{n+1}$ $n=1,2,\ldots,$, then from (3) we have

$$P_f(\bigcup_{n=1}^{\infty} A_n) = \bigvee_r P_r(\bigcup_{n=1}^{\infty} A_n) = \bigvee_r (\mathrm{Lim}P_r(A_n))$$

$$= \bigvee_r (\bigvee_n P_r(A_n)) = \bigvee_n (\bigvee_r P_r(A_n)) = \mathrm{Lim}P_f(A_n)$$

That is to say that P_f is continuous from blow. Similarly, the continuity from above can be obtained. The proof terminate.

Theorem 2.

Let $f\in M^+$,$P(X)<\infty$ and P be fuzzy additive. Then P_f is also fuzzy additive.

Proof. Let $E,F\in F(X)$, $E\cap F =\phi$ and $r\geq 0$.

Then $P_r(E \cup F) = r * P(\{f \geq r\} \cap (E \cup F)) = r * P((\{f \geq r\} \cap E) \cup (\{f \geq r\} \cap F))$
$= r * (P(\{f \geq r\} \cap E) \vee P(\{f \geq r\} \cap F))$
$= (r * P(\{f \geq r) \cap E)) \vee (r * P(\{f \geq r) \cap F))$
$= P_r(E) \vee P_r(F)$

Hence $P_f(E \cup F) = \bigvee_r P_r(E \cup F) = \bigvee_r (P_r(E) \vee P_r(F))$
$= (\bigvee_r P_r(E)) \vee (\bigvee_r P_r(F)) = P_f(E) \vee P_f(F)$

and this completes the proof.

Theorem 3.

Let f M^+ ,$P(X)<\infty$ and P is null-additive. Then P_f is also null-additive.

Proof. Be similar to the proof of theorem 2.

Theorem 4.

Let $f \in M^+$, $P(X)<\infty$ and P is above autocontinuous.Then P_f is also above autocontinuons.

Proof. Let $A, B_n \in F(X)$, $A \cap B_n = \phi$ $(n=1,2,\ldots)$, $P(B_n) \to 0$ $(n \to \infty)$

Then $P_f(A \cup B_n) = \bigvee_r P_r(A \cup B_n) = \bigvee_r P(\{f \geq r\} \cap (A \cup B_n))$
$= \bigvee_r P((A \cap \{f \geq r\}) \cup (B_n \cap \{f \geq r\}))$

Noticing that

$(A \cap \{f \geq r\}) \cap (B_n \cap \{f \geq r\}) = \phi$, $P(B_n) \to 0$ and P is above autocontinuous

we have

$P((A \cap \{f \geq r\}) \cup (B_n \cap \{f \geq r\})) \to P(A \cap \{f \geq r\})$ $(n \to \infty)$.

From above arguments and $P(X)<\infty$, we obtain

$P_f(A \cup B_n) \to P_f(A)$ $(n \to \infty)$ which completes the proof.

Theorem 5.

Let $\{f_n, f\} \subset M^+$,$P(X)<\infty$ and f_n converge to f on X uniformly. Then

$P_{f_n}(E) \to P_f(E)$ $(n \to \infty)$ for every $E \in F(X)$.

References

[1] Massimo SQUILLANTE and Aldo VENTRE, Fuzzy measure and convergence, Fuzzy Sets and Systems, 25(1988),251-257.

[2] Minghu HA and Xizhao WANG, On the convergence of fuzzy measure, J. of Hebei University, (in Chinese) 5(1989), 79-88

[3] D.Ralescu and G.Adams,The fuzzy integral,J.Math.Anal.Appl.,75(1980),562-570.