NOTES ON CONVEX DECOMPOSIITION OF BOLD FUZZY EQUIVALENCE RELATIONS

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Geometric properties of the convex set of bold fuzzy equivalence relations are studied in connection with a problem of decomposition into convex combination of max-min equivalence relations. Two particular combinatorial conditions of decomposability are proposed.

Keywords : extremal points, affine dimension, t-trans-convexity

1. NOTATIONS

Y - arbitrary set; $|Y|$ - cardinality of Y;

if Y is a subset of linear space, $\hat{Y}$ is convex hull of Y;

dually, if Y is a convex subset of linear space, $\check{Y}$ denotes the set of all extremal points of Y (non-decomposable in convex combination of two different points in Y);

$\tilde{\mathcal{F}}(Y)$ - set of all fuzzy subsets of Y;

$\mathbf{dim}(\cdot)$ - topological (linear, affine) dimension;

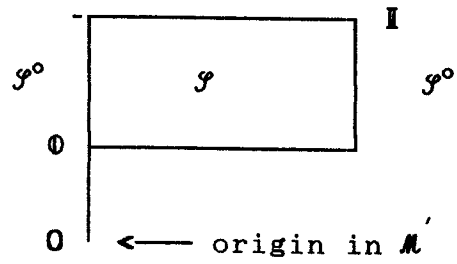
finite X, $|X|=n$ - support of all fuzzy/crisp binary relations (**FRs**/**BRs**);

$\vee$, $\wedge$ - max-min operations; $\dot{\wedge}$ - bold intersection;

superscript $^{\circ}$ is used to denote crisp objects;

$\mathcal{M}'$ - linear space of all symmetric real-valued $n{\times}n$ matrices with constant diagonal ($R\in \mathcal{M}' \leftrightarrow (r_{ii} \underset{i}{\equiv} \alpha \;\&\; r_{ij} \underset{i,j}{\equiv} r_{ji})$; more accurately, $\mathcal{M}'_n$- for brevity, subscript n is omitted in all designations);

$\mathcal{S}$ - set of all reflexive and symmetric **FRs**, considered both as $\vee$, $\wedge$ lattice with minimal element $\mathbb{O} = \begin{vmatrix} 1 & & 0 \\ \cdot & 1 & \cdot \\ 0 & & 1 \end{vmatrix}$, maximal element $\mathbb{I} = \begin{vmatrix} 1 & 1\ldots & 1 \\ \cdot & \cdot\cdot & \cdot\;\cdot \\ 1 & 1\ldots & 1 \end{vmatrix}$, and as $N = \dfrac{n(n-1)}{2}$ - dimensional affine cube $[\mathbb{O},\mathbb{I}]$, imbedded in $(N+1)$-dimensional linear space $\mathcal{M}'$;



$\|\cdot\|_1$ - Hemming norm in $\mathcal{S}$: $\|R\|_1 = \sum_{i<j} e_{ij}$ (only upper triangle of a matrix is used in all considerations);

$\rho_1(E,Y) = \inf_{R\in Y} \|E-R\|_1$ - Hemming distance between a relation $E \in \mathcal{S}$ and a subset $Y \subseteq \mathcal{S}$;

$V(E,1)$ – 1-neighborhood of E in Hemming metric, that is, intersection of $\mathcal{S}$ with Hemming sphere of radius 1, centered in $E \in \mathcal{S}$;

$\mathcal{S}^o$ - crisp reflexive&symmetric relations (vertexes of $\mathcal{S}$);

$\mathcal{E}$ - set of all traditional ($\vee$-$\wedge$) fuzzy equivalence relations;

$\mathcal{B}$ - set of all bold ($\vee$-$\wedge$) fuzzy equivalence relations;

$\mathcal{E}^o = \mathcal{B}^o$ - lattice of all crisp equivalence relations (partitions of X); $\{ij\}$ - atoms of the lattice $\mathcal{E}^o$, that is, partitions with single non-trivial equivalence class $\{ij\}$; note that $\{ij\}$'s form all closest to $\mathbb{O}$ vertexes of $\mathcal{S}$ (in other words, vertexes of $V(\mathbb{O},1)$).

2. BASIC PROPERTIES OF $\hat{\mathcal{E}}$, $\mathcal{B}$

"Decomposition problem" for bold equivalences was initiated by the discovering of an inclusion $\hat{\mathcal{E}} \subseteq \mathcal{B}$.

PROPOSITION 0. Primary properties of $\mathcal{E}^o$, $\mathcal{E}$, $\hat{\mathcal{E}}$, $\mathcal{B}$.

(i) both $\hat{\mathcal{E}}$, and $\mathcal{B}$ are convex subsets of $\mathcal{S}$;

(ii) $\hat{\mathcal{E}} = \hat{\mathcal{E}^o}$ - any convex combination of fuzzy $\vee$-$\wedge$ equivalence relations can be represented as convex combination of crisp equivalences;

(iii) $\check{\mathcal{E}} = \mathcal{E}^o \subseteq \check{\mathcal{B}}$ - crisp equivalences form **all** extremal points of $\mathcal{E}$, which are also extremal points (**not all!**) of $\mathcal{B}$ ■

PROPOSITION 1. With $n \le 3$, $\hat{\mathcal{E}} = \mathcal{B}$ ■

With $n \ge 4$, $\hat{\mathcal{E}}$ and $\mathcal{B}$ **do not** coincide. So, the main problem is

(P1) Find decomposability criterion.

However, since $\hat{\mathcal{E}}$ and $\mathcal{B}$ are convex subsets of $\mathcal{S}$, several problems in the spirit of "convex analysis" can also be stated:

(P2) Investigate location of $\hat{\mathcal{E}}$ in $\mathcal{B}$;

(P3) Describe all extremal points of $\mathcal{B}$;

(P4) Find most distant from $\hat{\mathcal{E}}$ extremal points of $\mathcal{B}$.

We start with P2-**P3** to motivate possible approach to **P1**.

First, let us study dimensions of $\hat{\mathcal{E}}$, $\mathcal{B}$. Owing to inclusions $\hat{\mathcal{E}} \subseteq \mathcal{B} \subseteq \mathcal{S}$, there formally exist four variants:

(a) $\dim(\hat{\mathcal{E}}) < \dim(\mathcal{B}) < \dim(\mathcal{S})$;

(b) $\dim(\hat{\mathcal{E}}) < \dim(\mathcal{B}) = \dim(\mathcal{S})$;

(c) $\dim(\hat{\mathcal{E}}) = \dim(\mathcal{B}) < \dim(\mathcal{S})$;

(d) $\dim(\hat{\mathcal{E}}) = \dim(\mathcal{B}) = \dim(\mathcal{S})$.

PROPOSITION 2. (dimension of $\hat{\mathcal{E}}$, $\mathcal{B}$).

(i) Variant **(d)** is in force; hence, $\hat{\mathcal{E}}$ and $\mathcal{B}$ are "bodies" in $\mathcal{S}$.

(ii) $\hat{\mathcal{E}}$ contains unit neighborhood of $\mathbb{O}$ $V(\mathbb{O},1) = \{\mathbb{O} \hat{\bigcup} \{ij\}\}$; maximal cube, contained in this neighborhood, is $[\mathbb{O},C]$ (**C** - for the **center** of simplex $\{\{\hat{ij}\}\}$), $C = \frac{N-1}{N}\cdot\mathbb{O} + \frac{1}{N}\cdot\mathbb{I}$.

(iii) $\hat{\mathcal{E}}$ contains "central domain" of $\mathcal{B}$, i.e. a neighborhood of the "diagonal" $\{C_\alpha = \alpha\cdot\mathbb{O} + (1-\alpha)\cdot\mathbb{I}\}$; thus, for any C_α, $V(C_\alpha, \alpha/2) \subset \hat{\mathcal{E}}$.

(iv) $\mathcal{B}$ contains the whole cube $[\mathbb{O}, \frac{1}{2}\cdot\mathbb{O} + \frac{1}{2}\cdot\mathbb{I}]$,

3. EXTREMAL NON-DECOMPOSABLE POINTS OF $\mathcal{B}$

Let $i \in X$; consider all **FRs**, containing only the i-th nonzero row/column (except for the diagonal); say, for $i = 1$

$$E = \begin{vmatrix} 1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_2 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ \alpha_n & & & 1 \end{vmatrix}$$

These **FRs** form a (n-1)-dimensional cube, which is denoted by $\mathcal{S}^{i} = [\mathbf{0}, \bigvee_{j}\{ij\}]$.

Next, with $\mathbf{a} = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\} \in \tilde{\mathcal{F}}(X \setminus \{i\})$, set

$$E^{i}(\mathbf{a}) = \bigvee \alpha_j \cdot \{ij\}.$$

PROPOSITION 3 (particular family of extremal non-decomposable points of $\mathcal{B}$).

(i) With $3 \le \mathbf{m} \le \mathbf{n}-1$, centers of all **m**-dimensional sides of $\mathcal{S}^{i}$

$$C^{iJ} = E^{i}\left(\frac{1}{2}\chi_J\right)$$

($J \subseteq X \setminus \{i\}$, $\mathbf{m} = |J| \ge 3$) are extremal points of $\mathcal{B}$; none of these **FRs** belongs to $\hat{\mathcal{E}}$.

$$C^{iJ} = \left|\begin{array}{ccccccccc}
1 & 0 & \overbrace{1/2 & 1/2 & \ldots & 1/2}^{J} & 0 & \ldots & 0 \\
0 & 1 & & & & & & & \\
1/2 & & 1 & & & \mathbf{0} & & & \\
1/2 & & & 1 & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2 & & & & & 1 & & & \\
0 & & \mathbf{0} & & & & 1 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & & & & & & & 1
\end{array}\right|$$

(rows $1/2, 1/2, \ldots, 1/2$ of the first column marked J)

(ii) The distance (in Hemming metric) between a center C^{iJ}, and the set of all decomposable bold equivalences

$$\rho_1(C^{iJ}, \hat{\mathcal{E}}) = \frac{\mathbf{m}}{2} - 1$$

(iii) Projection of C^{iJ} on $\hat{\mathcal{E}}$ is **not** included in $\mathcal{S}^{i}$; it contains all **FRs** in $\hat{\mathcal{E}}$, represented as

$$E = \sum_{j \in J} \lambda_j \{ij\} + \sum_{j,k \in J} \lambda_{jk} \{ijk\} ,$$

with λ_j, λ_{jk} satisfying two conditions:

$$\sum_{j} \lambda_j + \sum_{j,k} \lambda_{jk} = 1 \qquad \mathbf{(1)}$$

$$(\forall\, j)\left(\lambda_j + \sum_{k} \lambda_{jk} \le 1/2\right) \qquad \mathbf{(2)}$$

Extremal points of the projection of C^{iJ} on $\hat{\mathcal{E}}$ contain two families of **FRs**:

$$\frac{\{ik\}+\{il\}}{2} = \left|\begin{array}{cccccccc}
 & & \overset{k}{} & & \overset{J}{} & \overset{l}{} & & \\
1 & 0 & 1/2 & 0 & ..1/2..0 & & 0 & ...0 \\
0 & 1 & & & & & & \\
1/2 & & 1 & & \mathbf{0} & & & \\
0 & & & 1 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2 & & & & 1 & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & & & & 1 & & \\
0 & & & & & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & \mathbf{0} & & & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & & & & & & 1
\end{array}\right|
\begin{array}{l} \\ \\ \\ \leftarrow k \\ \leftarrow J \\ \\ \leftarrow l \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

(1/2 - exactly on two places in J), and

$$\frac{\{ikl\}+\{is\}}{2} = \left|\begin{array}{cccccccc}
 & & \overset{k}{} & \overset{l}{} & \overset{J}{} & \overset{s}{} & & \\
1 & 0 & 1/2 & 1/2 & ..1/2..0 & & 0 & ...0 \\
0 & 1 & 1/2 & & & & & \\
1/2 & 1/2 & 1 & & \mathbf{0} & & & \\
1/2 & & & 1 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2 & & & & 1 & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & & & & 1 & & \\
0 & & & & & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & \mathbf{0} & & & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & & & & & & & 1
\end{array}\right|
\begin{array}{l} \\ \\ \\ \leftarrow k \\ \leftarrow l \quad J \\ \\ \leftarrow s \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

(iv) For **n=4**, $\mathscr{E}^{\circ}\cup\{C^{iJ}\}$ exhausts all extremal points of $\mathscr{B}$, so that C^{iJ}s represent "most non-decomposable" bold equivalences ■

Proof of **(i)** is based on several simple propositions.

Denote by $\mathscr{E}^i = \mathscr{E}\cap\mathscr{S}^i$, $\hat{\mathscr{E}}^i = \hat{\mathscr{E}}\cap\mathscr{S}^i$, $\mathscr{B}^i = \mathscr{B}\cap\mathscr{S}^i$ sets of all traditional fuzzy equivalences, convex combinations of these equivalences and bold fuzzy equivalences, contained in $\mathscr{S}^i$. Let $\mathbf{a} = \{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n\} \in \tilde{\mathscr{F}}(X\setminus\{i\})$; set $E^i(\mathbf{a}) = \bigvee\alpha_j\cdot\{ij\}$. Clearly, $\|E^i(\mathbf{a})\|_1 = \|\mathbf{a}\|_1$.

LEMMA 1 (characterization of $\hat{\mathscr{E}}^i$). $\hat{\mathscr{E}}^i = \{\, E^i(\mathbf{a}) \mid \|\mathbf{a}\|_1 \le 1 \,\}$ ■

LEMMA 2 (characterization of $\mathscr{B}^i$).

$$\mathscr{B}^i = \{\, E^i(\mathbf{a}) \mid (\forall\, k,l \in X\setminus\{i\})(k\neq l \Rightarrow \alpha_k\wedge\alpha_l=0)\} \ \blacksquare$$

LEMMA 3. For all $E^1(a) \in \mathfrak{B}^1$, $\|a\|_1 \le \frac{1}{2}|supp(a)|$; equality is only for $a = (1/2,\dots,1/2)$ ■

Proof of (ii), (iii) is complicated, though routine.

For (iv), straightforward search of all extremal points of $\mathfrak{B}$ was done using special software tools ■

4. t-TRANS-CONVEXITY

Returning to P1, one can set a question: in what terms must we formulate decomposability criterion? We cannot expect the "symmetric polynomial" answer: indeed,

$$C^{1\{2,3,4\}} = \begin{vmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{vmatrix}$$

is non-decomposable, whereas

$$\begin{vmatrix} 1 & 1/2 & 1/2 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

contains the same number of 1/2's and is represented as $1/2\cdot\mathbb{0}+1/2\cdot\{123\} \in \hat{\mathcal{E}}$. Hence, any symmetric polynomial $\pi(\{e_{ij}\})$ does not distinguish between these two relations, $\pi(C^{1\{2,3,4\}}) = \pi(1/2\cdot\mathbb{0}+1/2\cdot\{123\})$.

Attempting to find answer in "algebraic" style, let us characterize $\hat{\mathcal{E}}$, and $\mathfrak{B}$ in terms of **t**-norms. Denote by $\mathcal{E}_t$ set of all $\bigvee$-**t**-equivalences, that is, reflexive, symmetric and $\bigvee$-**t**-transitive **FRs** (thus, $\mathcal{E} = \mathcal{E}_\wedge$, $\mathfrak{B} = \mathcal{E}_{\dot\wedge}$). Call a **t**-norm **t′** **t-trans-convex** iff $\hat{\mathcal{E}}_t \subseteq \mathcal{E}_{t'}$. Trans-convexity is described in a simple way.

PROPOSITION 4. t' is **t-trans-convex** iff for any $x, y \in [0,1]^2$, $\lambda \in [0,1]$,

$$t'(\lambda x+(1-\lambda)y) \le \lambda t(x)+(1-\lambda)t(y)$$

or, equivalently, $t'(\{\hat{x,y}\}) \le \{t(x)\hat{,}t(y)\}$ (in particular, $t' \le t$) ■

Clearly, if t' is itself convex, $t'(\{\hat{x,y}\}) \le \{t'(x)\hat{,}t'(y)\}$, then, for any $t \ge t'$ (that is, for any **t**-norm in the interval $[t',\wedge]$), t' is **t-trans-convex**. Within this approach, Lukasiewicz's **t**-norm has one more characterization.

PROPOSITION 5. $\dot{\wedge}$ is maximal convex **t**-norm ■

Define a **trans-convex hull** of any $\mathcal{E}_t$

$$\mathcal{K}(\mathcal{E}_t) = \bigcap_{t' \text{ is } t\text{-trans-convex}} \mathcal{E}_{t'}$$

(obviously, $\hat{\mathcal{E}}_t \subseteq \mathcal{K}(\mathcal{E}_t)$).

PROPOSITION 6. $\mathcal{K}(\mathcal{E}) = \mathcal{B}$; even more, for any $t \in [\dot{\wedge}, \wedge]$, $\mathcal{K}(\mathcal{E}_t) = \mathcal{B}$ ■

This is, in fact, one more unsuccessful attempt to describe decomposability conditions - for any collection **T** of **t**-norms, $\hat{\mathcal{E}}$ cannot be represented in the form $\bigcap_{t' \in T} \mathcal{E}_{t'}$.

5. COMBINATORIAL DECOMPOSABILITY CONDITIONS

So, let us try to find more "combinatorial" conditions. Let $E \in \mathcal{B}$. Denote by $\mathcal{Q}(E)$ the set of all crisp equivalences, not exceeding (that is, contained in) crisp relation $E_{>0}$:

$$\mathcal{Q}(E) = [\Phi, E_{>0}] \cap \mathcal{E}^{\circ},$$

by $\mathcal{C}(E)$ - the set of all coverings of $E_{>0}$ by of crisp equivalences (except for the atoms $\{ij\}$), contained in $\mathcal{Q}(E)$:

$$\mathcal{C}(E) = \{\ \mathfrak{X} \subseteq \mathcal{Q}(E)\ |\ (\text{no } E' \in \mathfrak{X} \text{ is atom})\ \&\ (\ \bigvee_{E' \in \mathfrak{X}} E' = E_{>0}\ )\ \}$$

With any $\mathfrak{X} \in \mathcal{C}(E)$, define **FR** $M(\mathfrak{X})$ as "arithmetical mean" of all **CR**'s in $\mathfrak{X}$:

$$M(\mathfrak{X}) = \sum_{E' \in \mathfrak{X}} E' \ /\ |\mathfrak{X}|.$$

Using these notations, a necessary, and a sufficient condition of decomposability can be formulated.

PROPOSITION 7 (a necessary condition for decomposability).

$$E \in \hat{\mathcal{E}} \Rightarrow (\ \exists\ \mathfrak{X} \in \mathcal{C}(E)\ )(|\mathfrak{X}|(\|M(\mathfrak{X})\|_1 - 1) \geq \|E\|_1 - 1)$$

(in particular, this condition is satisfied for any **E** with $\|E\|_1 \leq 1$) ■

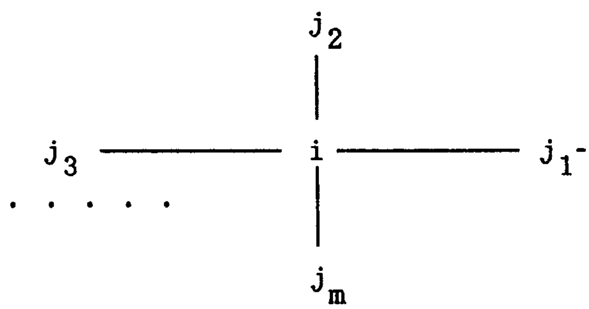
PROPOSITION 8 (a sufficient condition for decomposability).

$$\frac{M(\mathfrak{X})}{\|M(\mathfrak{X})\|_1 - 1} \subseteq \frac{E}{\|E\|_1 - 1} \Rightarrow E \in \hat{\mathcal{E}}$$

(here, both $\frac{M(\mathfrak{X})}{\|M(\mathfrak{X})\|_1-1}$, and $\frac{E}{\|E\|_1-1}$ are real-valued relations - not necessarily **FRs**, since their elements may exceed **1**; nevertheless, inclusion $\subseteq$ is used in the same meaning as in fuzzy case) ∎

Both the **PROPOSITION 7**, and the **PROPOSITION 8** can be written in a simpler fashion. However, using $M(\mathfrak{X})$ has certain advantages, emphasizing the "probabilistic" character of $\hat{\mathcal{E}}$ - any $E \in \hat{\mathcal{E}}$ can be considered as expected value of certain probability distribution on $\mathcal{E}^o$. In these terms, $M(\mathfrak{X})$ is the center of the simplex of all distributions with a given support $\mathfrak{X} \subseteq \mathcal{E}^o$.

Decomposability condition in **PROPOSITION 7** is closely related to the above example given in **PROPOSITION 3**. Indeed, for $J \subseteq X\backslash\{i\}$, $|J|=m\geq 3$, $(C^{iJ})_{>0}$ is a **m-star**



Clearly, $\mathcal{Q}(E^i(a)) = \{ \{ij\} \mid j \in J \}$, and $\mathcal{E}(E^i(a))=\emptyset$, so that left-hand expression in the inequality, given in **7**, is zero, which means $\|E^i(a)\|_1 \leq 1$ (cf. **LEMMA 1**).

In fact, all previous results are due to "small", or "rarefied" **FRs**. Can we reduce an arbitrary bold equivalence to the rarefied one (with $E_{>0}$ being considerably less than $\mathbb{I}$). Define $e_\wedge = \wedge e_{ij}$, and suppose $e_\wedge > 0$ ($\Leftrightarrow E_{>0} = \mathbb{I}$). Obviously, if $E \in \mathfrak{B}$ and $E_{>0} = \mathbb{I}$ then $E_0 = (E - e_\wedge \cdot \mathbb{I})/(1-e_\wedge) \in \mathfrak{B}$.

However, this subtraction doesn't preserve decomposability (that is, $E \in \hat{\mathcal{E}}$ must not imply $E_0 \in \hat{\mathcal{E}}$). A simple counter-example is for **n=5**, $E^{i(2)} \triangleq \frac{1}{6} \sum_{j,k} \{ijk\}$; thus, for $i = 1$,

$$E^{1(2)} = \begin{vmatrix} 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 1/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & 1 & 1/6 \\ 1/2 & 1/6 & 1/6 & 1/6 & 1 \end{vmatrix} \text{, and}$$

$$E_0=(E-1/6\cdot I)/(5/6) = \begin{vmatrix} 1 & 2/5 & 2/5 & 2/5 & 2/5 \\ 2/5 & 1 & 0 & 0 & 0 \\ 2/5 & 0 & 1 & 0 & 0 \\ 2/5 & 0 & 0 & 1 & 0 \\ 2/5 & 0 & 0 & 0 & 1 \end{vmatrix}$$ is non-decomposable.

There exist other possibilities - say, both E and E_0 may well be non-decomposable. For instance, let $i \in X$, $J \subseteq X\backslash\{i\}$, $|J|=m$, $\varepsilon<\frac{\frac{m}{2}-1}{N-m}$. Set $C^{iJ} = (1-\varepsilon)\cdot C^{iJ}+\varepsilon\cdot I$. Clearly, $(C_\varepsilon^{iJ}) = C^{iJ}$, and

$$\|C^{iJ}-C_\varepsilon^{iJ}\|_1 = (N-m)\cdot\varepsilon < \frac{m}{2} - 1$$

It follows from **PROPOSITION 3** (i), (ii), that none of C_ε^{iJ}, $(C_\varepsilon^{iJ})_0$ is decomposable).