THE ENTROPY OF THE Q-F-DYNAMICAL SYSTEM

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In this paper we shall investigate the properties of the system of fuzy subsets and of the state on this system, that enable to define the notion of the entropy. D. Markechová introducted (in [1]) this notion of the F - dynamical system, where the operations of intersection and union are defined by Zadeh (see [2]). We replace these operations in the following way:

$$(f \vee g)(t) = \min\{ f(t) + g(t), 1 \}$$

$$(f \wedge g)(t) = f(t).g(t)$$

for every $t \in \mathscr{X}$.

A complement of fuzzy subset f is a fuzzy subset $f^{\perp}$ such that $f^{\perp}(t) = 1 - f(t)$, for every $t \in \mathscr{X}$.

The fuzzy subsets f, g are orthogonal, and we write $f \perp g$, iff $f(t) \leq 1 - g(t)$ for every $t \in \mathscr{X}$.

Definition 1. Let $\mathscr{X}$ be a non - empty set and $\mathbb{M} \subset [0, 1]^{\mathscr{X}}$ such that

(i) if $1(t) = 1$ for any $t \in \mathscr{X}$, then $1 \in \mathbb{M}$

(ii) if $f, g \in \mathbb{M}$ and $f \leq g$, then $g - f \in \mathbb{M}$

(iii) if $f_n \in \mathbb{M}$, $n = 1,2, \ldots$ and $f_i \perp f_j$ for $i \neq j$, then

$$\bigvee_{n=1}^{\infty} f_n \in \mathbb{M}$$

(iv) if $f, g \in \mathbb{M}$, then $f \wedge g \in \mathbb{M}$.

Let $m : \mathbb{M} \to [0, 1]$ be a mapping satisfying the following conditions:

(v) $m(1) = 1$

(vi) $m\left(\bigvee_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} m\ (f_n)$, for any sequence $(f_n)_{n\in\mathbb{N}}$

such that f_i is orthogonal to f_j for $i \neq j$.

The trinity $(\mathcal{X}, \mathbb{M}, m)$ will be called a Q-F-quantum space.

Definition 2. Let $\mathcal{A} = \{f_1, \ldots, f_n\}$, $f_i \in \mathbb{M}$, $i = 1, \ldots, n$ be a finite system such that

(i) $f_i \perp f_j$ for $i \neq j$, $i, j = 1, \ldots, n$

(ii) $\sum_{i=1}^{n} f_i = 1$,

then $\mathcal{A}$ is called an orthogonal resolution of the unit.

We define the entropy $H_m(\mathcal{A})$ of a resolution $\mathcal{A}$ in the state m by

$$H_m(\mathcal{A}) = -\sum_{i=1}^{m} m\ (f_i) \log m\ (f_i).$$

Lemma 3. If $\mathcal{A} = \{f_1, \ldots, f_n\}$ and $\mathcal{B} = \{g_1, \ldots, g_m\}$ are orthogonal resolutions of the unit, then

$$\mathcal{A} \cup \mathcal{B} = \{\ f_i \wedge g_j\ ;\ f_i \in \mathcal{A} \text{ and } g_j \in \mathcal{B}\ \}$$

is an orthogonal resolution of the unit, too.

Theorem 4. Let $\mathcal{A}$ and $\mathcal{B}$ be orthogonal resolutions of the unit. Then there holds :

$$H_m\ (\mathcal{A} \cup \mathcal{B}\ ) \leq H_m(\mathcal{A}) + H_m(\mathcal{B}).$$

Definition 5. Let $(\mathcal{X}, \mathbb{M}, m)$ be a Q-F-guantum space and $T : \mathcal{X} \to \mathcal{X}$ be a maping such that

(i) if $f \in \mathbb{M}$, then $f \circ T \in \mathbb{M}$

(ii) $m\ (f \circ T) = m\ (f)$.

The quadruple $(\mathcal{X}, \mathbb{M}, m, T)$ will be called a Q-F-dynamical system.

Lemma 6. Let $(\mathcal{X}, \mathbb{M}, m, T)$ be a Q-F-dynamical system and $\mathcal{A} = \{f_1, \dots, f_n\}$ be a orthogonal resolution of the unit. Then $T(\mathcal{A}) = \{ f_1 \circ T, \dots, f_n \circ T \}$ is a orthogonal resolution of the unit, too.

We define $T^n(\mathcal{A}) = T\,[T^{n-1}(\mathcal{A})]$ for $n = 2, 3, \dots$.
By the preceding lemma we obtain that $T^n(\mathcal{A})$ is a orthogonal resolution of the unit.

Theorem 7. Let $(\mathcal{X}, \mathbb{M}, m, T)$ be a Q-F-dynamical system and $\mathcal{A}$ be an orthogonal resolution of the unit. Then there holds

$$H_m\,[T^n(\mathcal{A})] = H_m(\mathcal{A}) \text{ for } n = 1, 2, \dots .$$

Lemma 8. Let $(\mathcal{X}, \mathbb{M}, m, T)$ be a Q-F-dynamical system and $\mathcal{A}$ be an orthogonal resolution of the unit. Then there exists

$$\lim_{n\to\infty} \frac{1}{n} H_m \left[\bigvee_{j=0}^{n-1} T^j(\mathcal{A}) \right] .$$

Definition 9. Let $(\mathcal{X}, \mathbb{M}, m, T)$ be a Q-F-dynamical system. Then we will call

$h(T) = \sup \{ h(T, \mathcal{A}); \mathcal{A}$ is a finite orthogonal resolution of the unit $\}$

the entropy of a Q-F-dynamical system, where

$$h(T,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_m \left[\bigvee_{j=0}^{n-1} T^j(\mathcal{A}) \right] .$$

REFERENCES

[1] Markechová, D.: The entropy on F-quantum spaces. Mathematica Slovaca 2, 1990, 177 - 190.

[2] Zadeh, L. A.: Proability measure on fuzzy events. J. Math. Anal. Appl. 23, 1968, 421 - 427.