TWO NOTES ON MEASURE THEORY IN FUZZY TOPOLOGICAL SPACES

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In the classical measure and probability theory there are many nice and important results related to the topological structures. Therefore it seems to be meaningfull to expect some similar results in fuzzy topological spaces ([1]). In this contribution we present two typical results of this provenience: a regularity theorem for measures in fuzzy topological spaces and the Alexandrov theorem on the σ-additivity of additive regular functions. Of course, since we are concerned in a quite special area, we are able to present our results in an abstract form without topological assumptions.

1. A regularity theorem for submeasures

Definition 1. By an F-σ-ring we shall call any family F of fuzzy subsets of a set X satisfying the following three conditions:

(i) If $0_X(t) = 0$ for every $t \in X$, then $0_X \in F$.

(ii) If $f, g \in F$, $g \le f$, then $f - g \in F$.

(iii) If $f_n \in F$ $(n=1,2,\dots)$, then $\bigvee_{n=1}^{\infty} f_n \in F$.

A mapping $\mu : F \to \langle 0,\infty)$ will be called a submeasure, if the following two conditions are satisfied:

(iv) If $f, g, h \in F$ and $f \le g + h$, then $\mu(f) \le \mu(g) + \mu(h)$.

(v) If $f_n \searrow 0_X$, then $\mu(f_n) \to 0$.

It is not difficult to show that the following properties are satisfied:

1. If $f_n \in F$ $(n=1,2,\dots)$, then $\bigwedge_{n=1}^{\infty} f_n \in F$.
2. $\mu(0_X) = 0$.
3. If $f \le g$, then $\mu(f) \le \mu(g)$ and $\mu(g) - \mu(f) \le \mu(g-f)$.
4. $\mu(f \vee g) \le \mu(f) + \mu(g)$ for every $f, g \in F$.
5. If $f_n \nearrow f$ ($f_n \searrow f$) and $f_n \in F$ $(n=1,2,\dots)$, $f \in F$, then

$$\mu(f_n) \nearrow \mu(f) \quad (\mu(f_n) \searrow \mu(f)).$$

6. $\mu(\bigvee_{n=1}^{\infty} f_n) \leq \sum_{n=1}^{\infty} \mu(f_n)$ for every $f_n \in F$ $(n=1,2,\ldots)$.

It is interesting that every probability measure in the sense of *Kôpka* ([2], see also [3], [4]) is a submeasure following Definition 1.

Now we shall define the regularity of an element $f \in F$ with respect to a submeasure μ and two subsets C and U of F which corresponds to compact, or open sets resp.

Definition 2. Let us assume that there are given non-ampty subsets $C, U \subseteq F$ satisfying the following conditions:

(i) If $f \in C$, $g \in U$, then $f - (f \wedge g) \in C$.

(ii) If $f \in C$, $g \in U$ and $f \leq g$, then $g - f \in U$.

(iii) If $f_n \in U$ $(n=1,2,\ldots)$, then $\bigvee_{n=1}^{\infty} f_n \in U$.

(iv). If $n \in N$, $f_i \in C$ $(i=1,2,\ldots,n)$, then $\bigwedge_{n=1}^{n} f_i \in C$.

An element $f \in F$ is called regular, if to every $\varepsilon > 0$ there are $g \in C$, $h \in U$ such that $g \leq f \leq h$ and $\mu(h - g) < \varepsilon$.

Proposition 1. if f and g are regular and $f \leq g$, then also $g - f$ is regular.

Proposition 2. If f_n are regular $(n=1,2,\ldots)$, then $\bigvee_{n=1}^{\infty} f_n$ is regular, too.

2. The Alexandrov theorem

Definition 3. A non-empty set C of fuzzy subsets of a set X forms a compact family, if for every sequence $(f_i)_i \subseteq C$ the following implication holds:

$$(\forall\ n \in N\colon \bigwedge_{i=1}^{n} f_i \neq 0_X) \Rightarrow \bigwedge_{i=1}^{\infty} f_i \neq 0_X .$$

Definition 4. Let F be a set of fuzzy subsets of a set, $\mu: F \to \langle 0,\infty)$. We say that

a) μ is additive, if $\mu(f) = \sum_{i=1}^{n} \mu(f_i)$, whenever $f \in F$, $f_i \in F$ $(i=1,2,\ldots,n)$ and $f = \sum_{i=1}^{n} f_i$;

b) μ is σ-additive, if $\mu(f) = \sum_{n=1}^{\infty} \mu(f_n)$, whenever $f \in F$, $f_i \in F$

$(n=1,2,\dots)$ and $f = \sum_{n=1}^{\infty} f_i$;

c) μ is compact, if there exists a compact family $C \subseteq F$ such that to every $f \in F$ there is $g \in C$, $g \leq f$ with $\mu(f - g) < \varepsilon$.

Proposition 3. If $\mu: F \rightarrow \langle 0,\infty)$ is an additive and compact mapping defined on an F-σ-ring, then μ is σ-additive.

References

[1] Chang, C.: Fuzzy topological spaces. J. Math. Anal. Appl. 1968, 182 - 190.

[2] Kôpka, F.: Probability measure in fuzzy sets. Busefal, this issue.

[3] Riečan, B.: On the fuzzy approach to some problems of measure theory. Busefal 27, 1986, 10 - 15.

[4] Riečan, B.: Notes to the fuzzy measure theory (in Slovak). In: Theory and applications of fuzzy sets IV - VII, L. Mikuláš 1989, 176 - 186.