A REPRESENTATION OF OBSERVABLES ON A TYPE I,II,III OF FUZZY QUANTUM POSET

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1. Introduction.

Let Ω be a non-void set and $M \subseteq [0;1]^{\Omega}$ such that:

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in \Omega$.

(ii) if $a \in M$, then $a^{\perp} := 1-a \in M$.

(iii) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$.

A couple (Ω,M) is said to be a type I , type II , type III of fuzzy quantum poset (we write briefly FQP) if M is closed with respect to a union of any sequence of fuzzy sets mutually orthogonal, fuzzy orthogonal, strongly orthogonal, resp..Where a,b are orthogonal, fuzzy orthogonal , strongly orthogonal and we write $a \perp b$, $a \perp_F b$, $a \perp_S b$ iff $a+b \leq 1$, $a \cap b \leq 1/2$, $a \cap b = 0$, resp. and $\overset{\infty}{\underset{n=1}{\cup}} a_n := \text{Sup}\{a_n ; n \geq 1\}$.

If M is closed with respect to any sequence of fuzzy sets of M then (Ω,M) is said to be an F-quantum space. (See [7]).

It is obvious that an F-quantum space is an F-quantum poset every type 1,2,3, and an FQP type i is type i+1, i = 1,2 but the converse is not true, in general. (See [5]).

Recall that a mapping X from σ-algebra of Borel sets $B(R)$ into M is said to be an observable if

(i) $X(E^c) = X(E)^{\perp}$;

(ii) $X(\overset{\infty}{\underset{i=1}{\cup}} E_i) = \overset{\infty}{\underset{i=1}{\cup}} X(E_i)$ for any $E, E_i \in B(R)$, i = 1,2,..

2. Compatibility.

Definition 1. Let (Ω,M) be a type I,II,III FQP. A non-void subset A of M is said to be a Boolean σ-algebra of (Ω,M) if:

(i) There are the minimal and maximal elements $0_A, 1_A \in A$ such that $a \cap a^{\perp} = 0_A$, $a \cup a^{\perp} = 1_A$ for any $a \in A$.

(ii) With respect to 0_A, 1_A, $\perp$, $\cap$ and $\cup$, A is a Boolean σ-algebra (in the sense of Sikorski [8]).

Let now, Ω be a non-void set, Φ be any subset of $[0;1]^{\Omega}$.
For any $a \in \Phi$ such that $a \cup a^{\perp} \in \Phi$, we put:

$$\Phi_a = \{b \in \Phi;\ b \cup b^{\perp} = a \cup a^{\perp}\}, \tag{2.1}$$

$$\Omega_a = \{\omega \in \Omega;\ a(\omega) \neq 1/2\}, \tag{2.2}$$

$$\Omega_a(b) = \{\omega \in \Omega_a:\ b(\omega) = (a \cup a^{\perp})(\omega)\} \tag{2.3}$$

$$= \{\omega \in \Omega_a:\ b(\omega) > 1/2\}, \text{ for any } b \in \Phi_a$$

$$Q_a^{\Phi}(b) = \{\Omega_a(b);\ b \in \Phi_a\}, \tag{2.4}$$

If A is a Boolean σ-algebra of a type I,II,III FQP (Ω,M) then $A \subseteq M_a$ for any $a \in A$. In the case of given FQP (Ω,M), Q_a stand for Q_a^M .

Theorem 2. The mapping $\Omega_a(.)$: $\Phi_a \to Q_a^{\Phi}$ defined via:
$b \to \Omega_a(b)$

fulfills the following statements:

(i) $\Omega_a(a \cup a^{\perp}) = \Omega_a$;

(ii) $\Omega_a(b^{\perp}) = \Omega_a - \Omega_a(b)$ for any $b, b^{\perp} \in \Phi_a$;

(iii) for any $b,c \in \Phi_a$, $\Omega_a(b) \subseteq \Omega_a(c)$ iff $b \le c$;

(iv) for any $b,c \in \Phi_a$, $b \perp c$ iff $\Omega_a(b) \cap \Omega_a(c) = \emptyset$;

(v) if $\{b_i\}_{i=1}^{\infty} \subseteq \Phi_a$ and $\bigcup_{i=1}^{\infty} b_i \in \Phi_a$ then $\Omega_a(\bigcup_{i=1}^{\infty} b_i) = \bigcup_{i=1}^{\infty} \Omega(b_a)$.

Moreover, if (Ω,M) is a type I,II then Q_a is an q-σ-algebra and $\Omega_a(.)$ is isomorphism from M_a onto Q_a.

If (Ω,M) is a type III FQP and $a \in M$ such that $a \cup a^{\perp} \in M$ then Q_a needs not an q-σ-algebra, in general, as we may convince on simple examples.

Corollary 3. Let (Ω,M) be a type III FQP and $a \in M$ such that $a \cup a^{\perp} \in M$. Then two following statement are equivalent:

(i) Q_a is an q-σ-algebra.

(ii) $\bigcup_{n=1}^{\infty} b_n \in M_a$, for any sequence $\{b_n\}_{n=1}^{\infty} \subset M_a$ such that $a_i \perp a_j$ for $i \neq j$.

In these case $\Omega_a(.)$ is an isomorphism. Moreover, for any $a \in M$ and $A \subseteq M_a$, A is Boolean σ-algebra of M iff $\Omega_a(A)$ is a Boolean sub-σ-algebra of Q_a.

Corollary 4. Let (Ω,M) be a type I,II FQP. Suppose A be a non-void subset of M such that $a \cap b \in A$ if $a,b \in A$, for example if $a,b \in A$ then either $a \le b$ or $b \le a$. Then there is a Boolean σ-algebra of M containing A iff $A \subseteq M_a$, $a \in A$ and A is said to be σ-commensurable.

Corollary 5. Let (Ω,M) be a type III FQP and $a \in M$ such that at least (i) or (ii) of Corollary 3 is fulfilled. Then $A \subseteq M_a$ is σ-commensurable if $b \cap c \in A$ for any $b,c \in A$.

3. Representation of observables.

Theorem 6. Let X be an observable on a type I,II,III FQP (Ω,M), and let Q be the set of all rational numbers. Denote, for any $r \in Q$,

$$B_X(r) = X((-\infty;r))$$

Then the system $\{B_X(r);\ r \in Q\}$ fulfills the following conditions:

$$\begin{aligned}
&\text{(i)}\quad B_X(s) \le B_X(t) \quad \text{if } s < t \ ;\ s,t \in Q \ ;\\
&\text{(ii)}\quad \bigcup_{r \in Q} B_X(r) = a \ ;\ \bigcap_{r \in Q} B_X(r) = a^{\perp};\\
&\text{(iii)}\quad \bigcup_{s < r} B_X(s) = B_X(r) \ ,\ r \in Q \ ;\\
&\text{(iv)}\quad B_X(r) \cup B_X(r)^{\perp} = a \ ,\ r \in Q \ ;
\end{aligned} \tag{3.5}$$

where $a = X(R)$, $a^{\perp} = X(\emptyset)$.

Conversely, let $\{ B(r) \ ;\ r \in Q \}$ be a system of fuzzy sets fulfills the conditions (i)-(iv) for some $a \in M$, where (Ω,M) is a type I,II or type III such that $\{B(r);\ r \in Q\}$ are σ-commensurable, then there is an unique observable X on (Ω,M) such that $B_X(r) = B(r)$ for any $r \in Q$ and $X(R) = a$.

It can be pointed out that the converse of the Theorem 6 is not true in the of type III FQP, in general , if system $\{ B(r) \ ;\ r \in Q \}$ is not σ-commensurable. (See [5]).

Corollary 7. Let (Ω,M) be a type III FQP , let Q be the set of all rational numbers. $\{B(r);\ r \in Q\}$ is a system of fuzzy sets from M such that:

(i) $B(r) \le B(s);\ r < s;\ r,s \in Q$;

(ii) $\bigcup_{r \in Q} B(r) = a \ ;\ \bigcap_{r \in Q} B(r) = a^{\perp};\ \bigcup_{r<s} B(r) = B(s);\ r,s \in Q$;

(iii) $B(r) \cup B(r)^{\perp} = a$ for any $r \in Q$;

(iv) $\bigcup_{n=1}^{\infty} b_n \in M$ for any sequence $\{b_n\}_{n=1}^{\infty} \subseteq M$ such that

$$b_n \cup b_n^{\perp} = a \ ,\ b_n \le b_m^{\perp},\ n \ne m.$$

Then there is an observable X on M such that $X((-\infty;r)) = B(r)$.

Theorem 8. Let (Ω,M) be a type I,II,III FQP, X be an observable on M, then there is a K(M)-measurable function $f : \Omega \to R$ such that

$$\{X(E) > 1/2\} \subseteq f^{-1}(E) \subseteq \{X(E) \ge 1/2\} \tag{3.6}$$

for any Borel set E.

Moreover, if g is any K(M)-measurable, real-valued function on Ω, then g fulfills (3.6) iff

$$\{\omega \in \Omega;\ f(\omega) \ne g(\omega)\} \subseteq \{X(\emptyset) = 1/2\}.$$

An example, which points out that the converse of Theorem 8 is not true for type II, in general, can be see in [4].

For any sequence $\{a_n\}_{n=1}^{\infty}$ fuzzy sets of a type I,II FQP, then there exist $1_K = \bigcap_{n=1}^{\infty} (a_n \cup a_n^{\perp}) \in M$ and $0_K = 1_K^{\perp} \in M$. But, if (Ω,M) is type II, then $a_n \cap 1_K \cup 0_K$ does not belong to M, in general, which entails the converse of Theorem 8 fails for type II , in general. However, we have a converse of Theorem 8 as following.

Theorem 9. Let (Ω,M) be a type II FQP such that

$$a_n \cap (\bigcup_{n=1}^{\infty} (a_m \cup a_m^{\perp})) \in M \tag{3.7}$$

for any $n \geq 1$ and any sequence $\{a_n\}_{n=1}^{\infty}$ of M. If $f : \Omega \to R$ is any K(M)-measurable function, then there exists an observable X of (Ω,M) with (3.6). If Y is any observable of (Ω,M) with (3.6), then $X(E) \perp_F Y(E^c)$ for any $E \in \mathcal{B}(R)$.

It is known that the condition (3.7) of Theorem 9 is always fulfilled in a type I of fuzzy quantum poset.

Corollary 10. Let (Ω,M) be a type III FQP such that for any sequence $\{a_n\}_{n=1}^{\infty} \subseteq M$:

(i) $1_K = \bigcap_{n=1}^{\infty} (a_n \cup a_n^{\perp})$, $0_K = 1_K^{\perp} \in M$;

(ii) $a_n \cap 1_K \cup 0_K \in M$, for any $n \geq 1$; **(3.8)**

(iii) $\bigcup_{n=1}^{\infty} b_n \in M$ for any sequence $\{b_n\}_{n=1}^{\infty} \subseteq M$ such that $b_n \cup b_n^{\perp} = 1_K$ for $n \geq 1$ and $b_n \leq b_m^{\perp}$ for $n \neq m$.

If $f: \Omega \to R$ is an K(M)-measurable function, then there is an observable X on M with (3.6).

Note that a type III with (3.8) needs not a type II and a type II with (3.7) needs not a type I, as we can see in [5].

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