PROBABILITY MEASURE IN FUZZY SETS

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In the present paper the notion of a probability measure on an F-σ-algebra and a probability distribution of an observable in the measure will be defined. These notions allow us to introduce the mean value of an observable on an F-σ-algebra.

As a fuzzy subset of a nonempty set X we mean any function $a \in [0,1]^X$. For every sequence $(a_n)_{n\in N} \subset [0,1]^X$ of fuzzy sets let us define the operations of intersection and union in the following way (see e.g. Zadeh [4]):

$$(\bigwedge_{n\in N} a_n)(t) = \inf_{n\in N} a_n(t), \quad (\bigvee_{n\in N} a_n)(t) = \sup_{n\in N} a_n(t), \quad t \in X.$$

A complement of a fuzzy subset a of X is a function $a^\perp \in [0,1]^X$, such that $a^\perp(t) = 1 - a(t)$, for every $t \in X$. A diference of fuzzy subsets b and a of a set X is the function $b \setminus a \in [0,1]^X$ such that, $(b \setminus a)(t) = b(t) - (a \wedge b)(t)$, $t \in X$. A fuzzy set $a \in [0,1]^X$ is said to be the fuzzy subset of a set $b \in [0,1]^X$ (which will be denoted as $a \le b$), iff $a(t) \le b(t)$ for every $t \in X$. The fuzzy sets $a, b \in [0,1]^X$ are said to be (i) orthogonal, and we write $a \perp b$, if $a \le b^\perp$; (ii) disjoint, if $(a \wedge b)(t) = 0$ for every $t \in X$.

Definition 1. Let $M \subset [0,1]^X$ be a system of fuzzy subsets of a nonempty set X fulfiled the following conditions:

(i) if $\underline{1}(t) = 1$, for every $t \in X$, then $\underline{1} \in M$;

(ii) $\bigvee_{n\in N} a_n \in M$, for every sequence $(a_n)_{n\in N} \subset M$;

(iii) $b \setminus a \in M$, for every $a, b \in M$.

Then the system M will be called F-σ-algebra.

Let us note that, if the condition $1/2 \notin M$ is not considered, then an F-σ-algebra is a special case of a soft σ-algebra [1].

Definition 2. Let $M \subset [0,1]^X$ be a F-σ-algebra. A probability measure on M is a map m: $M \to [0,1]$ such that:

(i) $m(\underline{1}) = 1$;

(ii) if $(a_n)_{n\in N} \subset M$ be a sequence of a fuzzy sets for which there is $a \in M$ such that $a(t) = \sum_{n\in N} a_n(t)$, $t \in X$, then $m(a) = \sum_{n\in N} m(a_n)$.

Properties of a probability measure m on F-σ-algebra M.

1. $m(a) + m(a^\perp) = 1$, for every $a \in M$.
2. if $a, b \in M$, $a \le b$, then $m(a) \le m(b)$;
3. $m(\bigvee_{n\in N} a_n) = \sum_{n\in N} m(a_n)$ for every sequence $(a_n)_{n\in N} \subset M$ whenever $a_i \wedge a_j \equiv 0$, for $i \ne j$.
4. If $(a_n)_{n\in N} \subset M$ is the sequence that $a_i \le a_{i+1}$, $i \in N$, $\bigvee_{n\in N} a_n = a$, then $m(a) = \lim_{n\to\infty} m(a_n)$.
5. A map m on M is a valuation, i.e. $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$.
6. If a map m is defined on soft σ-algebra, then m is not identical to the P-measure.
7. Let M be F-σ-algebra. Let $t_0 \in X$ be any point. Then the map m: $M \to [0,1]$, such that $m(a) := a(t_0)$, $a \in M$, is a probability measure on M.

Definition 3. Let M be F-σ-algebra. Let $\mathfrak{B}(R)$ be a Borel σ-algebra of the real line R. The map x: $\mathfrak{B}(R) \to M$ is said to be observable on M if the following conditions are fulfiled:

(i) $x(R) = \underline{1}$;

(ii) if $A, B \in \mathfrak{B}(R)$, $A \subseteq B$, then $x(B \setminus A) = x(B) \setminus x(A)$;

(iii) if $(A_n)_{n\in N} \subset \mathfrak{B}(R)$, $A_n \subseteq A_{n+1}$, $n \in N$, then $x(\bigcup_{n\in N} A_n) = \bigvee_{n\in N} x(A_n)$.

Properties of an observable x on F-σ-algebra M.

1. $x(A^C) = (x(A))^\perp$, for every $A \in \mathfrak{B}(R)$ $(A^C = R \setminus A)$.
2. If $A \subseteq B$, $A, B \in \mathfrak{B}(R)$, then $x(A) \le x(B)$.

3. If $A \cap B = \emptyset$, $A, B \in \mathfrak{B}(\mathrm{R})$, then $x(A) \perp x(B)$.
4. An observable x, in general, is not a σ-homomorphism [2], [3].
5. If a map x from Definition 3 is defined on a quantum logic [3], then x is a σ-homomorphism.

Corollary 4. Let M be F-σ-algebra, m be a probability measure on M, x be an observable on M. Then the map $m_x : \mathfrak{B}(\mathrm{R}) \to [0,1]$ defined as $m_x(E) := m(x(E))$, $E \in \mathfrak{B}(\mathrm{R})$, is the probability measure on σ-algebra $\mathfrak{B}(\mathrm{R})$.

Proof. Let $(A_i)_{i\in\mathrm{N}} \subset \mathfrak{B}(\mathrm{R})$ be a sequence of a pairwise disjoint elements. Let $(B_n)_{n\in\mathrm{N}} \subset \mathfrak{B}(\mathrm{R})$ be a sequence such that $B_n = \bigcup_{i=1}^{n} A_i$, $n \in \mathrm{N}$. It is not difficult to see, that $x(B_n) = \sum_{i=1}^{n} x(A_i)$. Because of $x(A_i) \perp x(A_j)$, for $i \neq j$, then $\bigvee_{n\in\mathrm{N}} (\sum_{i=1}^{n} x(A_i)) = \sum_{i\in\mathrm{N}} x(A_i)$ is an element of M. Hence we have

$$m_x(\bigcup_{i\in\mathrm{N}} A_i) = m(x(\bigcup_{i\in\mathrm{N}} A_i)) = m(x(\bigcup_{n\in\mathrm{N}} B_n)) = m(\bigvee_{n\in\mathrm{N}} x(B_n)) = m(\sum_{i\in\mathrm{N}} x(A_i)) =$$
$$= \sum_{i\in\mathrm{N}} m(x(A_i)).$$

Q.E.D.

Definition 5. Let x be an observable on F-σ-algebra M, let m be a probability measure on M. A mean value of x in the measure m is an integral $\mathrm{E}(x) = \int_{\mathrm{R}} t \, \mathrm{d}m_x$, if this integral exists and is finite.

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it follows directly from the definition 2. This fact enables to overcome the troubles arising in classical erosion and dilation of the greylevel images where the values can arise and decrease up to infinity.

4. Conclusion

The greylevel images can be represented by functions defined on Euclidean or digital plane or by fuzzy subsets of Euclidean or digital plane and then processed by morphological transformations. The representation by fuzzy subsets is more suitable because the greylevel values are in a closed interval while processing of the images in opposite to the former case which is convenient for computer implementation.

References

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g is a fuzzy set $f \ominus g: Z^2 \to I$ with the membership function

$$(f \ominus g)(x) = \sup\{\alpha \in I; g_x \subseteq_\alpha f\}$$

where g_x is a function g shifted by x (i.e. $g_x(y) = g(y-x)$) and $g_x \subseteq_\alpha f$ denotes the fact that g_x is a fuzzy subset of f in degree α (i.e. for every y $1-g_x(y)+f(y) \geq \alpha$).

The fuzzy dilation of fuzzy set f by a SFS g is a fuzzy set $f \oplus g: Z^2 \to I$ with the membership function

$$(f \oplus g)(x) = 1-((1-f) \ominus g^s)(x)$$

The definition of fuzzy erosion is a generalization of the definition in example 1. The definition of fuzzy dilation uses a duality property as in the example 2.

For computing of the membership values more convenient expressions can be used which are shown in the following proposition:

Proposition 1: Let f be an image and g a SFS. Then the following equalities hold:

$$(f \ominus g)(x) = \min_{y \in Z^2}\{\min\{f(x+y)-g(y)+1, 1\}\}$$
$$(f \oplus g)(x) = \max_{y \in Z^2}\{\max\{f(x-y)+g(y)-1, 0\}\}$$

Proof: The proof follows directly from definition 2.

The following proposition shows that fuzzy erosion and fuzzy dilation are erosion and dilation in sense of definition 1:

Proposition 2: a) $\bigcup_i f_i \oplus g = \bigcup_i (f_i \oplus g)$

b) $\bigcap_i f_i \ominus g = \bigcap_i (f_i \ominus g)$

Proof:

a) $(\bigcup_i f_i \oplus g)(x) = \max_{y \in Z^2}\{\max\{\max_i\{f_i(y)\}+g(x-y)-1, 0\}\} =$

$= \max_i\{\max_{y \in Z^2}\{\max\{f_i(y)\}+g(x-y)-1, 0\}\}\} = \bigcup_i (f_i \oplus g)(x)$

b) the proof is dual to the proof of a)

From the proposition 1 it can be seen that fuzzy erosion and fuzzy dilation differ from classical erosion and dilation of greylevel images by adding resp. subtracting number 1 and by holding the values in bounded interval I as set of E^2. Then the transformation $d_B(X) = \{x \in E^2; B_{-x} \cap X \neq \emptyset\}$ is a dilation and $e_B(X) = \{x \in E^2; B_x \subseteq X\}$ is an erosion [1,2,3,4]. Here B_x is a set B shifted by vector $x \in E^2$, i.e. $B_x = \{b+x; b \in B\}$. The set B is called a structuring element. These transformations are the first studied morphological transformations [5] and were applied in processing of the binary images.

<u>Example 2:</u> Let L be a lattice of all real-valued functions defined on subsets of E^2 with the pointwise supremum and infimum. It is assumed that these functions have in the points of E^2 in which are not defined value $-\infty$ and let g is in L defined on a compact subset of E^2. Then the transformation $d_g(f) = \sup_{h \in D_g} \{f(x-h)+g(h)\}$ is a dilation and the transformation $e_g(f) = \inf_{h \in D_g} \{f(x+h)-g(h)\}$ an erosion [2,3]. The erosion and the dilation have the following duality property: $d_g(f) = -(e_{g^s}(-f))$ where $g^s(x)=g(-x)$. The function g is called a structuring function. These dilation and erosion are used in processing of the greylevel images.

3. *Fuzzy erosion and fuzzy dilation*

In this section the lattice L is the set of all fuzzy sets in Z^2 with the membership lattice $I=\{0, \frac{1}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1\}$ with the pointwise suprema and infima. The structuring function is also considered as the fuzzy set with membership function $Z^2 \to I$ and it will be called structuring fuzzy set (SFS). It will be assumed that SFS has a finite support i.e. a subset for which the value of membership function is greater than zero. Based on these notions fuzzy erosion and fuzzy dilation are defined by means of fuzzy subsets as follows:

<u>Definition 2:</u> Let $f: Z^2 \to I$ and $g: Z^2 \to I$ is a structuring fuzzy set. Then the fuzzy erosion of a fuzzy set f by a SFS