MULTIPLE FUZZY REASONING APPROACH TO FUZZY MATHEMATICAL PROGAMMING PROBLEMS

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We suggest solving fuzzy mathematical programming problems via the use of multiple fuzzy reasoning techniques. We show that our approach gives Buckley's solution [1] to possibilistic mathematical programs when the inequality relations are understood in possibilistic sense.

Keywords: Compositional rule of inference, multiple fuzzy reasoning, fuzzy mathematical programming, possibilistic mathematical programming, fuzzy implication

1. Preliminaries

Yager [5] introduced an approach to making inferences in a knowledge-based system which uses mathematical programming techniques. In this paper we apply multiple fuzzy reasoning (MFR) techniques to solve mathematical programming problems with fuzzy parameters. We show that our approach yields Buckley's solution [1] to possibilistic mathematical programs, when the inequality relations are understood in possibilistic sense.
We shall use the following inference rules:

The compositional rule of inference

Antecedent 1:	x and y have property W
Fact:	x has property P
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Consequence:	y has property Q

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where P and Q are fuzzy sets, W is a fuzzy relation and Q is obtained by the sup-min composition of P and W, i.e.

$$Q(y)=(P\circ W)(y)=\sup_x \min\{P(x),W(x,y)\}.$$

The generalized modus ponens (GMP)

Antecedent 1:	if x is A then y is B
Fact:	x is A'
Consequence:	y is $B'=A'\circ(A\to B)$

where $\to$ denotes a fuzzy implication operator and

$$B'(y)=\sup_x \min\{A'(x),\ A(x)\to B(y)\}$$

We say that the generalized modus ponens (under the implication operator $\to$) satisfies the

(i) fundamental property, if $B=A\circ(A\to B)$,

(ii) total indetermination property, if $(\text{not } A)\circ(A\to B)=\text{unknown}$

(iii) subset property, if $A'\circ(A\to B)=B,\ \forall A'\subseteq A$.

MFR scheme

Antecedent 1:	x and y have relation W_1
	...
Antecedent m:	x and y have relation W_m
Fact:	x has relation P
Consequence:	y has relation Q

where

$$Q=P\circ \min_{i=1,\dots,m} W_i,$$

i.e.

$$Q(y)=\sup_x \min\{P(x),\ \min_{i=1,\dots,m} W_i(x,y)\}$$

2. The new approach

Consider the fuzzy mathematical programming (FMP) problem

$$\begin{aligned} & g(\tilde{c},x) \longrightarrow \tilde{\max} \\ & f_1(\tilde{a}_1,x) \le \tilde{b}_1 \\ & \qquad \dots \\ & f_m(\tilde{a}_m,x) \le \tilde{b}_m \end{aligned} \qquad \text{(P1)}$$

where $\tilde{c}=(\tilde{c}_1,\dots,\tilde{c}_k)$ and $\tilde{a}_1=(\tilde{a}_{i1},\dots,\tilde{a}_{in_i})$ are vectors of fuzzy quantities (i.e. fuzzy sets of the real line R), $\tilde{b}_i$ is a fuzzy quantity, $x=(x_1,\dots,x_n)$ is a vector of decision variables, $g(\tilde{c},x)$ and $f_i(\tilde{a}_i,x)$ are defined by Zadeh's extension principle, and the inequality relation $\le$ is defined by a certain fuzzy relation.

Now, by using FMR techniques, we shall determine a fuzzy quantity, $\tilde{\max}$, which satisfies the inequality

$$g(\tilde{c},x) \le \tilde{\max},\ \forall x \in R^n,$$

under the constraints $f_i(\tilde{a}_i,x) \le \tilde{b}_i$, $i=1,\dots,m$.

We consider the inequality relation $\le$ as a fuzzy relation on R (which is defined by the decision-maker) and for every $x \in R^n$ determine $\tilde{\max}_x$ from the following MFR scheme

Antecedent 1:	$f_1(\tilde{a}_1,x) \le \tilde{b}_1$
	...
Antecedent m:	$f_m(\tilde{a}_m,x) \le \tilde{b}_m$
Fact:	$g(\tilde{c},x)$
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Consequence:	$\tilde{\max}_x$

where, according to the MFR inference rule,

$$\tilde{\max}_x = g(\tilde{c},x) \circ \min_{i=1,\dots,m} (f_i(\tilde{a}_i,x) \le \tilde{b}_i).$$

It is clear that $\tilde{max}_X$ is a fuzzy set realizing the inequality

$$g(\tilde{c},x) \le \tilde{max}_X$$

under the premises $f_i(\tilde{a}_i,x) \le \tilde{b}_i$, $i=1,\dots,m$.

Finally, we define the solution, $\tilde{max}$, of the FMP problem (P1) as

$$\tilde{max} = \sup_X g(\tilde{c},x) \circ \min_{i=1,\dots,m} (f_i(\tilde{a}_i,x) \le \tilde{b}_i)$$

It is clear that $\tilde{max}$ is a fuzzy set realizing the inequality

$$g(\tilde{c},x) \le \tilde{max},\ \forall x \in X,$$

under the premises $f_i(\tilde{a}_i,x) \le \tilde{b}_i$, $i=1,\dots,m$.

3. Relation with Buckley's solution

In this section we show that our approach yields Buckley's solution [1] to possibilistic mathematical programs, when the inequality relations are understood in possibilistic sense.
Consider the FMP problem:

$$Z=g(\tilde{c},x) \longrightarrow \tilde{max}$$

$$f_1(\tilde{a}_1,x) \le \tilde{b}_1$$

$$\dots$$

$$f_m(\tilde{a}_m,x) \le \tilde{b}_m$$

where the solution, $\tilde{max}$, is defined by

$$\tilde{max} = \sup_X g(\tilde{c},x) \circ \min_{i=1,\dots,m} (f_i(\tilde{a}_i,x) \le \tilde{b}_i)$$

Lemma 3.1. Let $\le$ be defined by

$$(f_i(\tilde{a}_i,x) \le \tilde{b})(u,v) = \begin{cases} \mathrm{Poss}[f_i(\tilde{a}_i,x) \le \tilde{b}] & \text{if } u=v \\ 0 & \text{otherwise} \end{cases}$$

then

$$(\tilde{max})(v) = \mathrm{Poss}[Z=v],\ \forall v \in \mathbb{R},$$

where Poss[Z=v], the possibility distribution of the objective function Z, is defined by [1,2]

$$\mathrm{Poss}[Z=v] = \sup_x \min\{\, g(\tilde{c},x)(v), \min_{i=1,\dots,m} \mathrm{Poss}[f_i(\tilde{a}_i,x) \le \tilde{b}_i]\}.$$

Really, from the definition of inequality relation we have

$$(\tilde{\max})(v)= \sup_x \sup_u \min\{\, g(\tilde{c},x)(u), \min_{i=1,\dots,m} (f_i(\tilde{a}_i,x) \le \tilde{b}_i)(u,v)\}$$

$$(\tilde{\max})(v)= \sup_x \sup_{u=v} \min\{\, g(\tilde{c},x)(u), \min_{i=1,\dots,m} \mathrm{Poss}[f_i(\tilde{a}_i,x) \le \tilde{b}_i]\}$$

$$(\tilde{\max})(v)= \sup_x \min\{\, g(\tilde{c},x)(v), \min_{i=1,\dots,m} \mathrm{Poss}[f_i(\tilde{a}_i,x) \le \tilde{b}_i]\}$$

which proves the lemma.

Remark 3.1. It can be shown that our solution concept (under well-chosen inequality relations and objective function) coincides with those ones suggested by Delgado et al [3], Ramik and Rimanek [4] and Zimmermann [6].

4. Illustrations

Consider the fuzzy linear programming (FLP) problem:

$$\langle \tilde{c},x\rangle \longrightarrow \tilde{\max}$$

$$\langle \tilde{a}_1,x\rangle \le \tilde{b}_1$$

$$\dots$$

$$\langle \tilde{a}_m,x\rangle \le \tilde{b}_m$$

where $\langle \tilde{c},x\rangle=\tilde{c}_1x_1+\dots+\tilde{c}_nx_n$, $\langle \tilde{a}_i,x\rangle=\tilde{a}_{i1}x_1+\dots+\tilde{a}_{in}x_n$ and the maximizing fuzzy set $\tilde{\max}$ is defined by

$$\tilde{\max} = \sup_x \langle \tilde{c},x\rangle \circ \min_{i=1,\dots,m} (\langle \tilde{a}_i,x\rangle \le \tilde{b}_i).$$

We illustrate our approach on simple FLP problems, where the inequality relation ≤ is defined by the Gödel implication:

$$x \leq y = x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

i.e.

$$(\langle \tilde{a}_i, x\rangle \leq \tilde{b}_i)(u,v) = \begin{cases} 1 & \text{if } \langle \tilde{a}_i, x\rangle(u) \leq \tilde{b}_i(v) \\ \tilde{b}_i(v) & \text{otherwise,} \end{cases}$$

Example 1.

The FLP problem	The adequate GMP scheme
$x\tilde{a} \longrightarrow \tilde{max}\, x$	Antecedent 1: $x\tilde{a} \leq \tilde{b}$
$x\tilde{a} \leq \tilde{b}$	Fact: $x\tilde{a}$
	Consequence: $\tilde{max}\, x = (x\tilde{a}) \circ (x\tilde{a} \leq \tilde{b})$,

since the Gödel implication satisfies the fundamental property of the GMP, we have

$$\tilde{max}\, x = \tilde{b}$$

the classical analogy,

$$\left.\begin{array}{c} xa \longrightarrow \max \\ xa \leq b,\ x \in R \end{array}\right\} \quad x_{opt} = b/a \quad \text{and} \quad x_{opt}a = b \quad (a \neq 0)$$

Example 2.

The FLP problem	The adequate GMP scheme
$x(\text{not } \tilde{a}) \longrightarrow \tilde{max}\, x$	Antecedent 1: $x\tilde{a} \leq \tilde{b}$
$x\tilde{a} \leq \tilde{b}$	Fact: $x(\text{not } \tilde{a})$
	Consequence: $\tilde{max}\, x = (x(\text{not } \tilde{a})) \circ (x\tilde{a} \leq \tilde{b})$

since the Gödel implication satisfies the total indetermination property of the GMP, we have

$$m\tilde{a}x = \text{unknown}$$

Example 3.

The FLP problem	The adequate GMP scheme
$x\tilde{a} \le \tilde{b}$	Antecedent 1: $x\tilde{a} \le \tilde{b}$
$x(\text{very } \tilde{a}) \longrightarrow m\tilde{a}x$	Fact: $x(\text{very } \tilde{a})$
	Consequence: $m\tilde{a}x = (x(\text{very } \tilde{a})) \circ (x\tilde{a} \le \tilde{b})$

since the Gödel implication satisfies the subset property of the GMP, we have

$$m\tilde{a}x = \tilde{b}$$

Example 4.

The FLP problem	The adequate MFR scheme
$x\tilde{a} \le \tilde{b}_1$	Antecedent 1: $x\tilde{a} \le \tilde{b}_1$
$x(\text{not } \tilde{a}) \le \tilde{b}_2$	Antecedent 2: $x(\text{not } \tilde{a}) \le \tilde{b}_2$
$x\tilde{a} \longrightarrow m\tilde{a}x$	Fact: $x\tilde{a}$
	Consequence: $m\tilde{a}x = \tilde{b}_1$

Example 5.

The FLP problem	The adequate MFR scheme
$x_1\tilde{a} \le \tilde{b}_1$	Antecedent 1: $x_1\tilde{a} \le \tilde{b}_1$
$x_2\tilde{a} \le \tilde{b}_2$	Antecedent 2: $x_2\tilde{a} \le \tilde{b}_2$
$x_1\tilde{a}+x_2\tilde{a} \longrightarrow m\tilde{a}x$	Fact: $x_1\tilde{a}+x_2\tilde{a}$
	Consequence: $m\tilde{a}x = \tilde{b}_1+\tilde{b}_2$

Concluding remarks. We have suggested solving fuzzy mathematical programming problems via the use of multiple fuzzy reasoning techniques. Our solution concept does not use the particular definition of inequality relations.

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