PROBABILITY LOGIC:SYNTAX

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1. Introduction

In the framework of fuzzy logic, in [3] I propose a semantic for probability logic in which the probabilistic models are the finitely additive probabilities defined in a Boolean algebra B. In this paper we propose a syntax for the probability logic and we prove a completeness theorem that relates this syntax with the semantics. Finally, a fuzzy sintax related to upper and lower probabilities is proposed.

2. A syntax for probability logic.

We define the "normalization function" $[\ ]:R\to[0,1]$ by setting $[x]=x$ if $0\leq x\leq 1$, $[x]=0$ if $x<0$ and $[x]=1$ if $x>1$. Moreover, for every $h, d, k\in N$, $h\geq d\geq k$, we call *h-d-k-rule* of inference the rule $r=(r',r'')$ where r' is defined in $D(h,d)=\{(\alpha_1,\dots,\alpha_h)/\ d(\alpha_1,\dots,\alpha_h)=d\}$ by

(2.1) $$r'(\alpha_1,\dots,\alpha_h) = C^k(\alpha_1,\dots,\alpha_h)$$

(where C^k is defined in [3]) and

(2.2) $$r''(x_1,\dots,x_h)= [(\textstyle\sum_i x_i-k+1)/(d-k+1)]$$

Also, we call *h-d-collapsing rule* the rule (c',c'') where c' is defined in $D(h,d)$ by setting $c'(\alpha_1,\dots,\alpha_h)=0$ and

$$c''(x_1,\dots,x_h)=\begin{cases}1 & \text{if } x_1+\dots+x_h > d\\ 0 & \text{otherwise.}\end{cases}$$

We call *probabilistic syntax* the fuzzy syntax such that:
- the set of formulas is **B** ;
- the logical axioms coincide with the classical logical axioms, i.e. with the fuzzy set $a:\mathbf{B}\rightarrow[0,1]$ such that $a(1)=1$, and $a(x)=0$ if $x\neq1$;
- the inference rules are the h-d-k-rules and the collapsing rules.

We call any theory on this syntax a *probabilistic theory*.

The next two propositions show the completeness of the probability logic.

Proposition 2.1 Given a probabilistic theory p, the following are equivalent

a) p is consistent ; b) $p(0)=0$; c) p is satisfiable.

As a consequence of Proposition 2.1, if v is consistent then it is possible to ignore both the collapsing rules and the normalization function in our proofs.

Proposition 2.2. Let p be a fuzzy subset of formulas, then p is a consistent probabilistic theory if and only if p is an envelope.

As a consequence, given a consistent map v, the probabilistic theory C(v) built up by (2.2) yields the envelope generated by v.

We have already observed that if p is a model of v and $\alpha\in\mathbf{B}$, then $[v(\alpha),1-v(\alpha)]$ is an interval approximation of $p(\alpha)$. We have also that $p(\alpha)$ is in $[C(v)(\alpha),1-C(v)(-\alpha)]$. Since $[C(v)(\alpha),1-C(v)(-\alpha)]$ is contained in $[v(\alpha),1-v(\alpha)]$, this means that the deductive apparatus of probability logic enables us to improve the initial interval approximation of p.

Several cases are possible; for example
- $[C(v)(\alpha),1-C(v)(-\alpha)]=\emptyset$ (v is inconsistent);
- $[C(v)(\alpha),1-C(v)(-\alpha)]=[0,1]$, that is $C(v)(\alpha)=C(v)(-\alpha)=0$, ($p(\alpha)$ is completely undetermined; v carries on no information about α);

- $[C(v)(\alpha),1-C(v)(-\alpha)]$ contains only a number, i.e. $C(v)(\alpha)+C(v)(-\alpha)=1$ ($p(\alpha)$ is completely determined by v).

As we will see in the next proposition, the last case happens whenever C(v) is a complete theory.

Proposition 2.3. Given a fuzzy subset p of formulas, the following are equivalent

a) p is a theory such that $p(\alpha)+p(-\alpha)=1$ for every formula α;

b) p is a probability ;

c) p is a complete probabilistic theory ;

d) p is a maximal consistent fuzzy subset of formulas;

e) p is consistent and $p(\alpha)+p(-\alpha)=1$ for every formula α.

The equivalence between b) and c) shows that probabilities are analogous to the complete theories in classical logic rather than to the theories. Then the concept of probability is not an adequate tool to give a foundation to probability logic and we have to consider the more general concept of envelope. On the other hand, it is a characteristic of the deductive apparatus of any logic to admit deductions under incomplete information, while the probabilities are related with complete information about a random situation.

Remark . Referring to Remark in [3] and by interpreting p as a betting function, it is possible to interpret Proposition 2.3 by asserting that the following are equivalent:

- p is a (finitely additive) probability ;
- every bet is reasonable but even a little augmentation of p gives rise to unreasonable bets;
- every bet is reasonable and a player cannot have sure winnings by betting on an event α and its opposite $-\alpha$ at the same time.

This latter condition characterizes the probabilities as those betting functions for which every bet is reasonable both from the player's and from the bank's point of view. This is in accordance with a subjective approach to probability theory.

3. Two examples.

Example 1. Usually, the probability $p(\alpha)$ of a sentence α is interpreted as the probability that a situation occurs in which α is true. Namely, if w is a set of elementary events and $p':\mathcal{P}(W)\rightarrow[0,1]$ is a probability, we set $p(\alpha)=p'(\{w\in W/\ w\models\alpha\})$. Now it seems rather natural to substitute the semantic relation $\models$ with weaker relations for example "we are able to recognize that α is true".

We can give a precise formulation of this by introducing a function $I:W\rightarrow\mathcal{P}(B)$ that we call "information function" and by interpreting $I(w)$ as the information available in the case that w occurs. Then the fuzzy subset of formulas $v:B\rightarrow[0,1]$ defined by

$$v(\alpha)=p(\{w\in W/\alpha\in I(w)\},$$

represents the probability that a situation occurs in which we are able to recognize that α holds. We have

- v consistent $\Longleftrightarrow$ $\forall w\in W$ $I(w)$ satisfies the finite intersection property (that is $I(w)$ is a consistent set of formulas)
- v probabilistic theory $\Longleftrightarrow$ $\forall w\in W$, $I(w)$ is a filter ($I(w)$ is a theory);
- v is a complete probabilistic theory $\Longleftrightarrow$ $\forall w\in W$, $I(w)$ is an ultrafilter ($I(w)$ is a complete theory).

Moreover,

- the theory $C(v)$ is defined by $C(v)(\alpha)=p(\{w\in W/\alpha\in T(w)\}$

where $T(w)$ denotes the filter (the theory) generated by $I(w)$;

- the probabilistic models m of v can be obtained by $m(\alpha)=p(\{w\in W/\alpha\in C(w)\}$ where, for every $w\in W$, $C(w)$ is a completion of $I(w)$.

Notice that the above construction generalizes the well known notion of belief function (see for example [9]). Indeed, if **B** is finite, $W=\{1,\ldots,n\}$ and for every $i\in W$ $I(i)$ is a filter generated by e_i then we have that $v(\alpha)=p(\{i\in W/e_i\leq\alpha\})=\sum\{m_i/e_i\leq\alpha\}$ where $m_i=p(\{i\})$. This proves that v is a belief function.

Example 2. Let R* be the class of the *interval numbers* i.e. the closed intervals of the real number set R, $p:\mathcal{P}(W)\to[0,1]$ a probability and $f:W\to R^*$ a map we call *non deterministic random variable* . For every subset Z of R we set

$$v(Z)=p(\{w\in W/f(w)\subseteq Z\}) \quad \forall Z\in\mathcal{P}(R),$$

i.e. v(Z) is the probability that ("every determination of") f satisfies Z. We have

- v is a probabilistic theory;
- v is complete $\Longleftrightarrow$ f is deterministic (single-valued).

Moreover the probabilistic models q of v can be obtained by

$$q(Z)=p(\{w\in W/g(w)\in Z\})$$

where g is a map such that $g(w)\in f(w)$.

4. Effectiveness.

In this section we will consider the question of the effectiveness of the proposed syntactical apparatus. Now, in [2] suitable notions of decidability and of recursive enumerability for fuzzy subsets are proposed. Namely, assume that **F** is a set such that there is an effective codification of **F** . Then we call *recursively enumerable* a fuzzy subset s of **F** such that $s(x)=\lim h(x,n)$ where $h:F\times N\to[0,1]$ is a computable map increasing with respect to n and with rational values. We say that s is *decidable* if both s and its complement $\sim s=1-s$ are recursively enumerable. Now, assume that F coincides with the Boolean algebra **B** and call *axiomatizable* a theory admitting a decidable fuzzy set of axioms. From Proposition 8.9 in [2] we get the following proposition.

Proposition 4.1 Every axiomatizable probabilistic theory is recursively enumerable and every axiomatizable complete probabilistic theory is decidable.

5. Refutation procedures

Recall that in classical logic we have that if Σ is a system of axioms and α a formula then

$$\Sigma \models \alpha \iff \Sigma \cup \{-\alpha\} \text{ is inconsistent.}$$

This property gives rise to deduction techniques called "refutation procedures". Likewise, in probability logic we can prove the following proposition where if $\delta \in L-\{0\}$, then $\{c\}^{\delta}$ is the "singleton" defined by

$$\{c\}^{\delta}(c)=\delta \text{ and } \{c\}^{\delta}(x)=0 \text{ if } x \neq c.$$

Proposition 5.1. For every consistent fuzzy subset v of formulas

$$C(v)(\alpha)=\bigvee\{\lambda / v \cup \{-\alpha\}^{1-\lambda} \text{ is inconsistent}\}. \tag{5.1}$$

This proposition together with the inconsistence condition given in Proposition 3.3 of [3], enables to give a very handful refutation system. Moreover it is possible to obtain a constructive formula to generate C(v) as follows.

Proposition 5.2. If $v:\mathbf{B}\rightarrow[0,1]$ is consistent

$$C(v)(\alpha)=\bigvee\left\{\frac{\Sigma v(\alpha_i) - d(\alpha_1,\dots,\alpha_h;\alpha)}{d(\alpha_1,\dots,\alpha_h)-d(\alpha_1,\dots,\alpha_h;\alpha)} \;/\; d(\alpha_1,\dots,\alpha_h)>d(\alpha_1,\dots,\alpha_h;\alpha)\right\} \tag{5.2}$$

where we have set $d(\alpha_1,\dots,\alpha_h;\alpha)=\min\{k\in N / C^k(\alpha_1,\dots,\alpha_h)\}\leq\alpha-1$.

If $\mathbf{B}$ is finite, (5.2) gives a simple algorithm to compute $C(v)(\alpha)$.

6. The sublogic of the upper and lower probabilities.

Perhaps it is of some interest to consider sub-syntaxes of the probability syntax obtained by confining ourselves to some inference rules. For example, if we consider the binary rules only, then we obtain a fuzzy syntax, that we call *ul-syntax* , strictly related with the upper and lower probabilities. Recall that an *upper and lower probability* is a pair (p_*,p^*) of functions from **B** into [0,1] such that:

$p^*(1)=1$, $p_*(1)=1$, $p^*(\alpha)=1-p_*(-\alpha)$ for every $\alpha\in\mathbf{B}$ and

(6.1) $$p_*(\alpha_1\vee\alpha_2)\geq p_*(\alpha_1)+p_*(\alpha_2) \quad ;$$

(6.2) $$p^*(\alpha_1\vee\alpha_2)\leq p^*(\alpha_1)+p^*(\alpha_2)$$

for every α and α_1 and α_2 such that $\alpha_1\wedge\alpha_2=0$ (see [5]).

Proposition 6.1. A pair (p_*,p^*) is an upper and lower probability if and only if p_* is a consistent theory on the ul-syntax and $p^*(\alpha)=1-p_*(-\alpha)$ for every $\alpha\in\mathbf{B}$.

Proposition 6.1 entails that the "intersection" of a family of upper and lower probabilities is an upper and lower probability and that the formula (2.2) in [3] enables us to build up the upper and lower probability "generated" by a given fuzzy subset of formulas.

It is not clear if an adequate semantics for the ul-syntax exists.

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