Optimal Average

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1. Intorduction

Let $(\Omega, \mathcal{F})$ be a measureable space. A function $p: \mathcal{F} \to [0, 1]$ is called optimal measure if it satisfies the following properties:

P1 $p(\emptyset) = 0$ and $p(\Omega) = 1$.
P2 p is M-additive, i.e. $p(B \cup E) = p(B) \vee p(E)$, B and $E \in \mathcal{F}$.
P3 p is continuous form above.

The triple $(\Omega, \mathcal{F}, p)$ will be called optimal measure space (cf. [1]).

Lemma 1 Any optimal measure is continuous from above.

Remark The collection $L = \{B \in \mathcal{F} : p(B) < p(\Omega)\}$ is a σ-ideal, and for every $B \in \mathcal{F}$,

$$p(B') = 1 - p(B) + p(B) \wedge p(B')$$

(where B' is the complement of B).

In the sequel, measurable sets (resp. functions) will be referred to as *events* (resp. *random variables*, abbreviated *r.v.*'s) and measurable simple functions as *discrete r.v.*'s.

2. The optimal average of nonnegative discrete r.v.'s

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be a nonnegative discrete r.v. (where $\chi(B)$ is the characteristic function of the event B). The quantity $I(s) = \bigvee_{i=1}^{n} b_i \chi(B_i)$ is called the *optimal average* of s, and for any event B, $I_B(s) = I(s\ \chi(B))$ is called the optimal average of s on B.

We have shown that the definition of the optimal average is a correct one (i.e. $I(s)$ does not depend on the decomposition of s).

Proposition 2

1. $I(\chi(B)) = p(B), B \in \mathfrak{F}$.
2. The functional $I(s)$ is a positively homogeneous, monotone increasing, sub-additive, and preserves the maximum operation.
3. $I_B(s) = 0$, if $p(B) = 0$.
4. $I_B(s) = I(s)$, if $p(B') = 0$.
5. $I_B(s) = \lim_{n\to\infty} I_{B_n}(s)$, whenever $(B_n) \subset \mathfrak{F}$ tends monotonically to B as $n\to\infty$.

3. *The optimal average of nonnegative r.v.'s*

Proposition 3 Let $f \geq 0$ be any bounded r.v. Then

$$\sup_{s\leq f} I(s) = \inf_{\bar{s}\geq f} I(\bar{s}),$$

where s and $\bar{s}$ denote nonnegative discrete r.v.'s.

Let $f \geq 0$ be a r.v. The quantity $Af = \sup_{0\leq s\leq f} I(s)$ will be referred to as the optimal average of f, and for any event B, $A_B f := A(f\chi(B))$ as the optimal average of f on B.

Proposition 4

1. The optimal average is a positively homogeneous, monotone increasing, sub-additive and a maximum operation preserving functional.
2. $A_B f = 0$ if $p(B) = 0$, and $A_B f = Af$ if $p(B') = 0$.

In comparison with the symbol $\int$ of the Lebesgue integral, we shall adopt the symbol $\mathfrak{l}$ to designate optimal average, i.e.,

$$Af = \mathfrak{l}_{\Omega} f\,dp, \quad A_B f = \mathfrak{l}_{B} f\,dp$$

where $f \geq 0$ is any r.v. and B any event.

Definition 1 A property will be said to hold almost surely (abbreviated a.s.) if the set of elements where it fails to hold is a set of optimal measure zero.

Proposition 5 (*Optimal Monotone Convergence*)

i) Let (f_n) be an increasing sequence of nonnegative r.v.'s and f its limit. Then $Af = \lim_{n\to\infty} Af_n$.

ii) Let (g_n) be a decreasing sequence of nonnegative r.v.'s and g its limit, with g_1 being bounded. Then $Ag = \lim_{n\to\infty} Ag_n$.

Lemma 6 (*Optimal Fatou*)

i) Let (f_n) be a sequence of nonnegative r.v.'s. Then

$$A(\liminf_{n\to\infty} f_n) \le \liminf_{n\to\infty} Af_n$$

ii) Let (g_n) be a uniformly bounded sequence of nonnegative r.v.'s. Then

$$\limsup_{n\to\infty} Ag_n \le A(\limsup_{n\to\infty} g_n)$$

Theorem 7 (*Optimal Dominated Convergence*) Let (f_n) be a uniformly bounded sequence of nonnegative r.v.'s, with f its a.s. limit. Then $Af = \lim_{n\to\infty} Af_n$.

Proposition 8 Let f be a nonnegative r.v.

i) $Af = 0$ if and only if $f = 0$ a.s.

ii) If $Af < \infty$, then $f < \infty$ a.s.

Proposition 9 (*Optimal Markov*) Let f be a nonnegative r.v. with finite optimal average. Then for every real number $x > 0$, $xp(f \ge x) \le Af$.

Proposition 10 Let f be a bounded nonnegative r.v. Then for any positive real number ε, there exists a positive real number δ such that $A_E f < \varepsilon$ whenever $p(E) < \delta$, $E \in \mathcal{F}$.

Proposition 11 Let $f \ge 0$ be a r.v., $Af < \infty$. If $b \le \frac{1}{p(B)} A_B f \le c$ for all event B, $p(B)>0$, where b and c are some given positive constants, then $b \le f \le c$ a.s.

Definition 2 We shall say that a sequence of r.v's (f_n) converges in optimal measure to a r.v. f, if for each constant $\varepsilon > 0$, $\lim_{n\to\infty} p(|f - f_n| \ge \varepsilon) = 0$ (abbreviated $f_n \underset{p}{\rightarrow} f$).

Theorem 12 (*Optimal Riesz*) Let $f_n \underset{p}{\rightarrow} f$. Then there exists a subsequence (f_{n_k}), $k \ge 1$, which converges to f a.s.

Definition 3 Let $f : \Omega \to \overline{\mathbf{R}}$ be any r.v. We shall say that

i) $f \in \mathcal{A}^{\infty}$, if $p(|f| \le b) = 1$ for some positive constant b.

ii) $f \in \mathcal{A}^{\alpha}$, if $A|f|^{\alpha} < \infty$, $\alpha \in [1, \infty)$.

For $\alpha \in [1, \infty]$, the functional

$$\|f\|_{\mathcal{A}^\alpha} = \begin{cases} \inf(b>0 : p(|f| \le b) = 1, & \text{if } f \in \mathcal{A}^\infty \\ \{A\,|f|^\alpha\}^{1/\alpha}, & \text{if } f \in \mathcal{A}^\alpha \text{ with } \alpha \in [1, \infty) \end{cases}$$

is a norm. (The corresponding Hölder (resp. Minkowski) inequality is obtained and is called *Optimal Hölder* (resp. *Optimal Minkowski*) *inequality*.)

Theorem 13 For $\alpha \in [1, \infty]$, the space $\mathcal{A}^\alpha$ endowed with the above norm is a Banach space.

4. The optimal Fubini theorem

Let $(\Omega_i, \mathcal{F}_i, p_i)$, $(i = 1, 2)$, be two optimal measure spaces and let us denote the smallest σ-algebra containing $\mathcal{F}_1 \times \mathcal{F}_2$ by $\mathcal{S} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$.
For each $\omega_1 \in \Omega_1$ (resp. $\omega_2 \in \Omega_2$) we define ω_1 (resp. ω_2) cross-section by
$E_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in E\}$ (resp. $E^{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in E\}$), where $E \in \mathcal{S}$.

Definition 4 Let f be any r.v. defined on $(\Omega_1 \times \Omega_2, \mathcal{S})$. For every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the functions
i) $f_{\omega_1} : \Omega_2 \to \overline{\mathbf{R}}$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$
respectively,
ii) $f_{\omega_2} : \Omega_1 \to \overline{\mathbf{R}}$ defined by $f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$
will be called ω_1-section, respectively ω_2-section of f.

Theorem 14 For every $E \in \mathcal{S}$, define the functions

$$m_E : \Omega_1 \to \overline{\mathbf{R}}_+ \text{ by } m_E(\omega_1) = p_2(E_{\omega_1})$$

and

$$m^E : \Omega_2 \to \overline{\mathbf{R}}_+ \text{ by } m^E(\omega_2) = p_1(E^{\omega_2})$$

Then
i) m_E is $\mathcal{F}_1$-measurable.
ii) m^E is $\mathcal{F}_2$-measurable.
iii) $\int_\Omega m_E\, dp_1 = \int_\Omega m^E\, dp_2$
Furthermore, define the function $p_1 \times p_2 : \mathcal{S} \to [0, 1]$ by

$$(p_1 \times p_2)(E) = \int_\Omega m_E \, dp_1 = \int_\Omega m^E \, dp_2.$$

Then $p_1 \times p_2$ is an optimal measure such that $(p_1 \times p_2)(B \times D) = p_1(B)p_2(D)$, for all $B \in \mathcal{F}_1$ and $D \in \mathcal{F}_2$.

Theorem 15 (*Optimal Fubini*) Let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1,2$, be two optimal measure spaces and let $f \in \mathcal{A}^1(\Omega_1 \times \Omega_2, \mathcal{S}, p_1 \times p_2)$ be any r.v. Then

1. The ω_1-section $|f_{\omega_1}| : \Omega_2 \to \overline{\mathbf{R}}_+$ belongs to $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$ almost surely on Ω_1.

The function $\varphi : \Omega_1 \to \overline{\mathbf{R}}_+$, defined by $\varphi(\omega_1) = \int_\Omega |f\omega_1| \, dp_2$, belongs to $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$.

2. The ω_2-section $|f_{\omega_2}| : \Omega_1 \to \overline{\mathbf{R}}_+$ belongs to $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$ almost surely on Ω_2.

The function $\psi : \Omega_2 \to \overline{\mathbf{R}}_+$, defined by $\psi(\omega_2) = \int_\Omega |f\omega_2| \, dp_1$, belongs to $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$.

3. Furthermore,

$$\int_{\Omega \times \Omega} |f| \, d(p_1 \times p_2) = \int_\Omega \left(\int_\Omega |f| \, dp_2 \right) dp_1 = \int_\Omega \left(\int_\Omega |f| \, dp_1 \right) dp_2$$

5. *Structure of optimal measures*

By (p-)atom we mean an event H, $p(H)>0$ such that whenever $B \in \mathcal{F}$, $B \subset H$, then $p(H)=p(B)$ or $p(B)=0$ (cf. [2]).

Definition 5 We shall say that a (p-)atom H is *decomposable* if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(B \setminus H)$. If no such subatom exists, we shall say that H is *indecomposable*.

Theorem 16 (*Fundamental Optimal Measure*) There exists a sequence of disjoint indecomposabel (p-)atoms $\mathcal{H}=\{H_1, H_2, \dots\}$, $\lim_{n \to \infty} p(H_n) = 0$, such that

$$p(B) = \bigvee_{n=1}^{\infty} p(B \cap H_n) = \max(p(H_n) : p(H_n \setminus B) = 0)$$

for any event B with $p(B)>0$.

Definition 6 The sequence $\mathcal{H} = \{H_1, H_2, \dots\}$ of disjoint indecomposable atoms, obtained in the Fundamental Optimal Measure theorem, will be called *(p-)generating system*.

Definition 7 Let $q : \mathcal{F} \to \mathbf{R}_+$ be a set function satisfying properties P1-P3, with the hypothesis $q(\Omega) = 1$ in P1 being replaced by the hypothesis $0 < q(\Omega) < \infty$. We shall say that q is *absolutely continuous* relative to p (abbreviated $q \ll p$) if $q(B) = 0$ whenever $p(B) = 0$, $B \in \mathcal{F}$.

Proposition 17 $q \ll p$ if and only if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $q(B) < \varepsilon$ whenever $p(B) < \delta$, $B \in \mathcal{F}$ (where q is defined as in Definition 7).

Theorem 18 (*Optimal Radon-Nikodym*) Let q be defined as in Definition 7. Suppose that $q \ll p$. Then there exists a unique r.v. $f \geq 0$ such that for every event B, $q(B) = \underset{B}{\iota} f\, dp$.

(This r.v. can be explicitly given and will be called the *Optimal Radon-Nikodym derivative* of q relative to p, and will be denoted by $\frac{dq}{dp}$.)

Let E be a fixed event, $p(E) > 0$. Consider the set function $p^* : \mathcal{F} \to [0, 1]$ defined by $p^*(B) = \frac{p(B \cap E)}{p(E)}$. Then p^* is an optimal measure and $p^* \ll p$. Moreover, $\frac{dp^*}{dp} = \frac{\chi(E)}{p(E)}$ p-a.s.

Definition 8 The function $p^*(B)$ will be called *conditional optimal measure* of B given E, and denoted by $p(B|E)$, i.e. $p(B|E) = \frac{p(B \cap E)}{p(E)}$.

Definition 9 Let $f \in \mathcal{A}^1$ be a r.v. and $E \in \mathcal{F}$, $p(E)>0$. The conditional optimal average of f given E is defined by $A(|f| \mid E) = \underset{\Omega}{\iota} |f|\, dp^*$.

Proposition 19 If $f \in \mathcal{A}^1$, then for every event E, $p(E)>0$, $A(|f| \mid E) = \frac{1}{p(E)} \underset{E}{\iota} |f|\, dp$.

Proposition 20 Let $\mathcal{H} = \{H_1, H_2, \ldots\}$ be a p-generating system and $f \in \mathcal{A}^1$. Then

$$A|f| = \bigvee_{n=1}^{\infty} \{A(|f| \mid H_n)\, p(H_n)\}$$

Especially, for any event B,

$$p(B) = \bigvee_{n=1}^{\infty} \{p(B \mid H_n)\, p(H_n)\}$$

Let $\mathcal{E} \subset \mathfrak{F}$ be a σ-algebra and $\mathcal{H}=\{H_1, H_2, \dots\}$ be a p-generating system.
By $\mathcal{E}$-measurable sub-generating system of $\mathcal{H}$, we mean an $\mathcal{H}^*=\{H_1^*, H_2^*, \dots\} \subset \mathcal{H}$ such that $\mathcal{H}^* \subset \mathcal{E}$ and for all $E \in \mathcal{E}$ with $p(E) > 0$, $p(E) = \max(p(H_n^*) : p(H_n^* \setminus E)=0)$.

Theorem 21 Let $\mathcal{E}$ be any sub-σ-algebra of $\mathfrak{F}$ and $0 \leq f \in \mathcal{A}^1$. Let $\mathcal{H}^*=\{H_1^*, H_2^*, \dots\}$ be an $\mathcal{E}$-measurable sub-generating system of $\mathcal{H}$. Then

$$g = \bigvee_{n=1}^{\infty} \{A(f \mid H^*_n)\, \chi(H^*_n)\}$$

is the unique nonnegative r.v. satisfying the following two conditions:
i) g is $\mathcal{E}$-measurable,
ii) $\int g\, dp = \int f\, dp$ for all $E \in \mathcal{E}$.

Definition 10 Let $\mathcal{E}$ and f be as in Theorem 21. The r.v. g defined there will be referred to as the *conditional optimal average* of f given $\mathcal{E}$, and will be denoted by $A(f \mid \mathcal{E})$.
Especially,
1. If $\mathcal{E} = \sigma(h)$, where h is a r.v., then $A(f \mid \sigma(h))$ will be called the conditional optimal average of f given h, and will be denoted by $A(f \mid h)$.
2. If $f = \chi(B)$ for an event $B \in \mathfrak{F}$, then $A(\chi(B) \mid \mathcal{E})$ will be called the *conditional measure* of B given $\mathcal{E}$, and will be denoted by $p(B \mid \mathcal{E})$.

Proposition 22 Let f and $g \in \mathcal{A}^1$ be two nonnegative r.v.'s, $b \in \mathbf{R}_+$, $\mathcal{E}$ and $\mathcal{E}_0=\{\emptyset, \Omega\}$ be two sub-σ-algebras of $\mathfrak{F}$. Then we have:
1. $A(f \mid \mathcal{E}) \geq 0$, $A(f \mid \mathcal{E}_0) = Af$.
2. $A(f \mid \mathcal{E}) = 0$ a.s. if $f = 0$ a.s.
3. $A(A(f \mid \mathcal{E})) = Af$.
4. $A(1 \mid \mathcal{E}) = 1$ a.s.
5. $A(bf \mid \mathcal{E}) = bA(f \mid \mathcal{E})$ a.s.
6. $A(f \vee g \mid \mathcal{E}) = A(f \mid \mathcal{E}) \vee A(g \mid \mathcal{E})$ a.s.
7. $A(f \mid \mathcal{E}) \leq A(g \mid \mathcal{E})$ a.s. if $f \leq g$ a.s.
8. $A(f \mid \mathcal{E}) = f$ a.s. if f is $\mathcal{E}$-measurable.

Theorem 23 Let $\mathcal{E}$ be a sub-σ-algebra of $\mathfrak{F}$.
1. Let (f_n) be an increasing sequence of nonnegative r.v's and f its limit with $Af < \infty$. Then

$$A(f \mid \mathcal{E}) = \lim_{n \to \infty} A(f_n \mid \mathcal{E}) \quad \text{a.s.}$$

2. Let (g_n) be a decreasing sequence of nonnegative r.v.'s and g its limit with $g_1 \leq b$ for some $b \in \mathbf{R}_+$. Then

$$A(g \mid \mathcal{E}) = \lim_{n \to \infty} A(g_n \mid \mathcal{E}) \quad \text{a.s.}$$

Lemma 24 Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$.

1. Let (f_n) be a sequence of nonnegative r.v.'s. Then

$$A(\liminf_{n \to \infty} f_n \mid \mathcal{E}) \leq \liminf_{n \to \infty} A(f_n \mid \mathcal{E}) \quad \text{a.s.}$$

2. Let (g_n) be a uniformly bounded sequence of nonnegative r.v.'s. Then

$$\limsup_{n \to \infty} A(g_n \mid \mathcal{E}) \leq A(\limsup_{n \to \infty} g_n \mid \mathcal{E}) \quad \text{a.s.}$$

Theorem 25 Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$, and (f_n) be a uniformly bounded sequence of nonnegative r.v.'s with f its a.s. limit. Then $\lim_{n \to \infty} A(f_n \mid \mathcal{E}) = A(f \mid \mathcal{E})$ a.s.

Proposition 26 Let $f \geq 0$, $g \geq 0$, be two r.v.'s, f and $fg \in \mathcal{A}^1$; and let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$. If g is $\mathcal{E}$-measurable, then $A(fg \mid \mathcal{E}) = gA(f \mid \mathcal{E})$ a.s.

References

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