The Countable Additivity of Set-valued Integrals and F-valued Integrals

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Abstract; In the paper, integrals of set-valued functions and F-valued functions are further investigated, the countable additivity of the integrals is shown.

keywords; set-valued integral, F-valued integral, countable additivity.

1. Introduction

In recent years, there are a lot of works on set-valued integrals such as convexity theorem, Dorminate convergence theorem, and so on [1, 2, 3,], but there is few in the additivity. The purpose of the paper is to show that integrals of set-valued functions have the property of countable additivity, then extended it to integrals of F-valued functions.

Let $(X, \mathcal{A}, \mu)$ be a complete finite measure space, R^n be the n-dim Euclidean space. Symbol $P(R^n)$ denotes the power set of R^n, co $\mathcal{K}$ denotes the family of nonempty compact convex subsets of R^n. The addition on $P(R^n)$ is defined as follow;

$$\sum_{n=1}^{\infty} Bn = \{ \sum_{n=1}^{\infty} b_n;\ b_n \in Bn\ (n \geq 1),\ \sum_{n=1}^{\infty} \| b_n \| < \infty \}$$

for $Bn \in P(R^n)$ $(n \geq 1)$. Where $\| \cdot \|$ is the Euclidean norm.

Further, we define $\|A\| = \mathrm{Sup}\ \{\|a\|, a \in A\}$ for $A \in P(R^n)$.

Let $\mathcal{F}(R^n)$ be the family of all fuzzy subsets in R^n. An element $\tilde{a} \in \mathcal{F}(R^n)$ is said to be a F-number if $a_\lambda = \{r \in R^n; \tilde{a}(r) \geqslant \lambda\} \in co\mathcal{K}$ for every $\lambda \in (0, 1]$, and we use $\mathcal{F}^*$ to denote the set of F-numbers. Define

$$(\sum_{n=1}^{\infty} \tilde{r}_n)(u) = \mathrm{Sup}\ \{\ \bigwedge_{n=1}^{\infty} \tilde{r}_n(u_n);\ u = \sum_{n=1}^{\infty} u_n\ ,\ \sum_{n=1}^{\infty} \|u_n\| < \infty\}$$

for $\tilde{r}_n \in \mathcal{F}^{*'}(n \geqslant 1)$.

A set-valued measure π is a countably additive mapping from to $P(R^n) - \{\phi\}$; a F-number measure $\tilde{\pi}$ is a countably additive mapping from $\mathcal{A}$ to $\mathcal{F}^*$, i.e

$$\pi(\bigcup_{n=1}^{\infty} An) = \sum_{n=1}^{\infty} \pi(An)$$

$$\tilde{\pi}(\bigcup_{n=1}^{\infty} An) = \sum_{n=1}^{\infty} \tilde{\pi}(An)$$

for every sequence An $(n \geqslant 1)$ of mutually disjoint elements of $\mathcal{A}$. A set-valued function $F; X \to P(R^n) - \{\phi\}$ is measurable if $GrF = \{(x, \omega) \in X \times R^n;\ \omega \in F(x)\} \in \mathcal{A} \times \mathrm{Borel}(R^n)$. A F-Set-valued function $\tilde{F}; X \to \mathcal{F}(R^n) - \{\phi\}$ is measurable if its λ-cutting set-valued function $F_\lambda(\cdot); x \to F_\lambda(x) = [F(X)]_\lambda$ is measurable for all $\lambda \in (0, 1]$. A set-valued function is integrably bounded if there exists an integrable function $h; X \to R^+$, s. t. $\|F(x)\| \leqslant h(x)$ for all $x \in X$. A F-set-valued function F is called a F-valued function if it is valued in $\mathcal{F}^*$. A F-valued function $\tilde{F}$ is integrably bounded if there exists an integrable function $h; X \to R^+$ s. t. $\mathrm{Sup}_{\lambda \in (0,1]} \|F_\lambda(x)\| \leqslant h(x)$ for all $x \in X$.

The integral of $F; X \to P(R^n) - \{\phi\}$ on $A \in \mathcal{A}$ is defined as;

$$\int_A F d\mu = \{\int_A f d\mu\ ;\ f \in S(F)\}$$

where $S(F)=\{f$ is integrable, $f(x)\in F(X)$ a.e. $x\in X\}$

The integral of a F-valued function on $A\in\mathcal{A}$ is defined as ,

$$(\int_A \tilde{F}d\mu)(u)=\mathrm{Sup}\{\lambda\in(0,1],u\in\int_A F_\lambda d\mu\}$$

where $u\in R$.

The paper is organized as follow. In section 2, we discuss the set-valued integral and obtain the theorem of countable additivity. In section 3, the integral of F-valued functions is investigated, the countable additivity is shown.

2. On set-valued integrals.

The main result of this section is the following theorem, and it is not mentioned by the authors in ref.,

Theorem 1. Let $F_i, X\to P(R^n)-\{\phi\}$ $(i\geq 1)$ be a sequence of measurable set-valued functions, let $\phi_i, X\to R^+(i\geq 1)$ be a sequence of measurable functions. If

i) $\| F_i(x)\| \leq \phi_i(x)$ for all $x\in X$;

ii) $\phi(x)=\sum_{i=1}^{\infty}\phi_i(x)$ is integrable.

Then

i) Set-valued function $F(\cdot), x\to \sum_{i=1}^{\infty} F_i(x)$ is integrably bounded;

ii) $S(F)=\sum_{i=1}^{\infty} S(F_i)$;

iii) $\int F d\mu=\sum_{i=1}^{\infty}\int F_i d\mu$.

Proof. It is clear that $\| F(x)\| \leq \sum_{i=1}^{\infty}\phi_i(x)=\phi(x)$, then i) is proved, Obviously, $S(F)\supset\sum S(F_i)$, hence, to prove ii) is to verify sufficiently that $S(F)\subset\sum S(F_i)$ For this aim, let $f\in S(F)$, we need to find a sequence $f_i, X\to R^n(i\geq 1)$. s. t . $f_i\in S(F_i)$, $f(x)=\sum_{i=1}^{\infty} f_i(x)$. Now the space $R^n\times R^n\cdots$ may be metrized

s. t. its topology is the usual product topology, we define a set-valued function $x \to G(x) = \{(u_1, u_2 \cdots), u_i \in Fi(x)\ (i \geq 1)\} \subset L^1(R^n)$ (the Banach space of sequence $\{u_i\} \subset R^n$, and $\| \{u_i\} \| = \sum_{i=1}^{\infty} \| u_i \| < \infty$), by the measurablity of $Fi(i \geq 1)$, it is easy to see that G is a measurable set-valued function (the definition of measurablity is same for a metric space). Further we define a continuous mapping $H: L^1(R^n) \to R^n$; $\{u_i\} \to \sum_{i=1}^{\infty} u_i$, then the mapping $x \to H^{-1}(f(x)) = \ker H + \{f(x)/2^i\}$ is a measurable set-valued function from X to $L^1(R^n)$. Since the intersection of two measurable set-valued functions is also measurable, and $f \in S(F)$, then the set-valued function $G(\cdot) \cap H^{-1}(\cdot)$ is nonempty measurable, therefore there exists an a.e measurable selection

$$x \to \{f_i(x)\},\ f_i(x) \in Fi(x)$$

$$\text{s.t } f(x) = \sum_{i=1}^{\infty} f_i(x), \text{ a.e } x \in X$$

Consequenly, ii) has been proved. iii) is a direct result of ii).

Corollary 1. If a measurable set-valued function F is integrably bounded, then for every sequence {An} of mutually disjoint elements of $\mathcal{A}$.

$$\int_{\bigcup_{n=1}^{\infty} A_n} F d\mu = \sum_{n=1}^{\infty} \int_{A_n} F d\mu$$

Proof. We first define $a \cdot F(x) = \{a \cdot y, y \in F(x)\}$, for $a \in R$. Then

$$\int F d\mu = \int X_A(x) F d\mu, \text{ where } X_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

Let $F_i(x) = X_{A_i}(x)F(x)$, $\phi_i(x) = X_{A_i}(x)\phi(x)$, it is easy to see the conditions of the theorem are satisfied, then the conclusion is held. (Q.E.D)

Corollary 2. If a measurable set-valued function F is integrably bounded, then

$$\pi(A) = \int_A F d\mu$$

is a set-valued measure.

Corollary 3. Let F_1, F_2 be two integrably boundd set-valued functions, a, $b \in R$. Then

$$\int (a \cdot F_1 + b \cdot F_2) d\mu = a \cdot \int F_1 d\mu + b \cdot \int F_2 d\mu.$$

3. On F-valued integrals

A F-valued function is the mapping from X to $\mathcal{F}^*$, the integral of F-valued functions was first given in [6], but the author didn't discuss the problem on additivity. Here we will deal with it.

Lemma 1. [4] Let $\{\tilde{a}_n\}$ be a sequence of F-numbers, $\sum_{n=1}^{\infty} \tilde{a}_n \in \mathcal{F}^*$, then

$$\left(\sum_{n=1}^{\infty} \tilde{a}_n\right)_\lambda = \sum_{n=1}^{\infty} a_{n\lambda}, \qquad \lambda \in (0, 1]$$

Lemma 2. [6] If $\tilde{F}: X \to \mathcal{F}^*$ is an integrably bounded and measurable F-valued function, then

$$\int \tilde{F} d\mu \in \mathcal{F}^*, \text{ and } \left(\int \tilde{F} d\mu\right)_\lambda = \int F_\lambda d\mu, \quad \lambda \in (0, 1]$$

Theorem 2. Let $\tilde{F}n$ $(n \geq 1)$ be a sequence of measurable F-valued functions, and ϕ_n: $X \to R^+$ $(n \geq 1)$ be a sequence of measurable functions, If

i). $\sup_{\lambda \in (0,1]} \| Fn_\lambda(x) \| \leq \phi_n(x), (x \in X)$;

ii) $\phi(x) = \sum_{n=1}^{\infty} \phi_n(x)$ is integrable;

iii). The mapping $\tilde{F}(\cdot), x \to \tilde{F}(x) = \sum_{n=1}^{\infty} \tilde{F}_n(x)$ is a F-valued function;

Then ,

i) $\tilde{F}$ is integrably bounded;

ii) $\int \tilde{F} d\mu = \sum_{n=1}^{\infty} \int \tilde{F}n d\mu$.

Proof. i) It is easy to see that, for $\lambda \in (0, 1]$ by lemma 1 we have $F_\lambda(x) = \sum_{n=1}^{\infty} F_{n\lambda}(x)$, then

$$\sup_{\lambda \in (0,1]} \| F_\lambda(x) \| \leq \sum_{n=1}^{\infty} \sup_{\lambda \in (0,1]} \|F_{n\lambda}(x)\| \leq \sum_{n=1}^{\infty} \phi_n(x) = \phi(x)$$

Consequently, i) is proved,

ii)By lemma 2, for $\lambda \in (0, 1]$, we have

$$(\int \tilde{F}(x) d\mu)_\lambda = \int F_\lambda(x) d\mu$$

$$= \sum_{n=1}^{\infty} \int F_{n\lambda}(x) d\mu \quad \text{(by theorem 1)}$$

$$= \sum_{n=1}^{\infty} (\int \tilde{F}n(x) d\mu)_\lambda$$

$$= (\sum_{n=1}^{\infty} \int \tilde{F}n(x) d\mu)_\lambda$$

Therefore $\int \tilde{F}(x) d\mu = \sum_{n=1}^{\infty} \int \tilde{F}n(x) d\mu$ (Q.E.D)

By the theorem, the following results can be obtained directly.

Corollary 4. Let $\tilde{F}: X \rightarrow \mathcal{F}^*$ be an integrably bounded and measurable F-valued function, then for every sequence $\{An\}$ of mutually disjoint elements of $\mathcal{A}$.

$$\int_{\bigcup_{n=1}^{\infty} A_n} \tilde{F} d\mu = \sum_{n=1}^{\infty} \int_{A_n} \tilde{F}n(x) d\mu$$

Corollary 5. Let $\tilde{F}: X \rightarrow \mathcal{F}^*$ be an integrably bounded and measurable F-valued function, then

$$\tilde{\pi}(A)=\int_A \tilde{F} d\mu$$

is a F-number measure.

For $a \in R^{+}$, define $a\cdot\tilde{b}$ as $(a\cdot\tilde{b})_\lambda = a\cdot\tilde{b}_\lambda$ $(\lambda\in(0,1])$, then

Corollary 6. Let $\tilde{F}_1, \tilde{F}_2: X\rightarrow \mathcal{F}^{*}$ be integrably bounded and measurable F-valued functions , a, $b\in R$. Then

$$\int (a\cdot\tilde{F}_1+b\cdot\tilde{F}_2)d\mu = a\cdot\int \tilde{F}_1 d\mu + b\cdot\int \tilde{F}_2 d\mu.$$

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