THE F-INTEGRAL OF F-FUNCTIONS AND R-N THEOREM OF F-NUMBER MEASURES

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The purpose of this paper is to define the f-integral of F-functions and investigate the relation between this F-integral and the F-number measures. The Radon-Nikodym theorem of F-number measures is also proved.

Keywords: F-function, F-number measure, F-integral, set-valued function, set-valued measure, Aumann's integral.

1. Preliminaries

Let X be a nonempty set; R^m the m-dimensional Euclidean space; $\mathbf{P}(R^m)$, $\mathbf{C}(R^m)$, $\mathbf{K}(R^m)$, $Co\mathbf{P}(R^m)$, $Co\mathbf{K}(R^m)$ and $\mathbf{F}(R^m)$ the family of all subsets, closed subsets, compact subsets, convex subsets, compact convex subsets and fuzzy subsets of R^m, respectively. Moreover, the classical measures mentioned in this paper are non-atomic measures.

Definition 1.1.[1] Let $(X, \mathbf{A})$ be a measurable space. A set-valued mapping $\Pi: \mathbf{A} \to \mathbf{P}(R^m)-\{\emptyset\}$ is called a set-valued measure, if $\forall \{A_i\} \subseteq \mathbf{A}$ and $A_i \cap A_j = \emptyset$ $(i \neq j) \Longrightarrow \Pi(\bigcup_{n=1} A_n) = \sum_{n=1}^{\infty} \Pi(A_n) \triangleq \{\sum_{n=1}^{\infty} r_n,\ r_n \in \Pi(A_n),\ n \geq 1,\ \sum_{n=1}^{\infty} \|r_n\| < \infty\}$. If $\Pi(A) \in \mathbf{K}(R^m)$ ($Co\mathbf{P}(R^m)$, $Co\mathbf{K}(R^m)$), $\forall A \in \mathbf{A}$, then Π is called compact (convex, compact convex) set-valued measure.

Definition 1.2.[3] A fuzzy number on R^m is a fuzzy subset $\mu \in \mathbf{F}(R^m)$ with the property: $\mu_\alpha \triangleq \{r \in R^m: \mu(r) \geq \alpha\} \subseteq Co\mathbf{K}(R^m)-\{\emptyset\}$ for $\forall \alpha \in (0, 1]$. We denote $\mathbf{F}^*(R^m)$ by the family of all fuzzy numbers on R^m.

For $\forall r \in R^m$ and $\forall u_n \in \mathbf{F}^*(R^m)$ $(n \geq 1)$, let $(\sum_{n=1}^{\infty} u_n)(r) \triangleq \sup\{\inf_{n \geq 1} \mu_n(r_n):$ $r = \sum_{n=1}^{\infty} r_n,\ r_n \in R^m,\ \sum_{n=1}^{\infty} \|r_n\| < \infty$.

Definition 1.3.[3] Let $(X, \mathbf{A})$ be a measurable space. A F-valued mapping $F: \mathbf{A} \longrightarrow \mathbf{F}^*(R^m)$ is called a F-number measure, if $\forall\{A_i\} \subseteq \mathbf{A}$ and $A_i \cap A_j = \emptyset$ $(\forall\, i \neq j) \Longrightarrow F(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} F(A_n)$.

Definition 1.4.[4] Let $(X, \mathbf{A})$ be a measurable space. A set-valued function $\pi: X \longrightarrow \mathbf{P}(R^m)$ is said to be measurable, if $\pi^{-1}(C) \triangleq \{x \in X: \pi(x) \cap C \neq \emptyset\}$ $\in \mathbf{A}$ for $\forall C \in \mathbf{C}(R^m)$.

Definition 1.5.[2] Let $(X, \mathbf{A}, v)$ be a finite measure space, $\pi: X \longrightarrow \mathbf{P}(R^m)$ a set-valued function. For $\forall A \in \mathbf{A}$, if $v(A) \neq 0$, then the Aumann's integral of π over A is defined as

$$(A)\ \int_A \pi dv \triangleq \left\{ \int_A g dv: g \in S(\pi) \right\}$$

Where $S(\pi) \triangleq \{g: X \longrightarrow R^m$ is integrable with respect to v, and $g(x) \in \pi(x)$ v-a.e.$\}$. If $(A) \int_A \pi dv \neq \emptyset$ then π is called A-integrable over A. If there exists a integrable (with respect to v) function $h: X \rightarrow \mathbf{R}$ such that $\|r\| \leq h(x)$ for $\forall x \in A, \forall r \in \pi(x)$, then π is said to be integrably bounded on A.

Proposition 1.1.[2] Let $\pi: X \longrightarrow Co\mathbf{P}(R^m) \cap \mathbf{C}(R^m)$ be a closed convex measurable set-valued function. If π is integrably bounded with respect to a finite measure v, then $(A) \int_A \pi dv \in Co\mathbf{K}(R^m) - \{\emptyset\}$, $A \in \mathbf{A}$.

2. F-measurable F-functions and F-integrals

Let $f: X \longrightarrow \mathbf{F}(R^m)$ be a F-function. For $\forall \alpha \in (0, 1]$, let $f_\alpha(x) \triangleq [f(x)]_\alpha = \{r \in R^m: [f(x)](r) \geq \alpha\}$, $\forall x \in X$; $[f^{-1}(u)]_\alpha = f_\alpha^{-1}(u_\alpha) \triangleq \{x \in X: f_\alpha(x) \cap u_\alpha \neq \emptyset\}$, $\forall u \in \mathbf{F}(R^m)$. Then $f_\alpha: X \longrightarrow \mathbf{P}(R^m)$ is a set-vaued function, and $f^{-1}: \mathbf{F}(R^m) \longrightarrow \mathbf{F}(X)$. Where $f^{-1}(u) = \bigcup_{\alpha \in (0,1]} \alpha . [f^{-1}(u)]_\alpha$ and $\mathbf{F}(X)$ is the family of all fuzzy subsets of X.

Definition 2.1. Let $\mathbf{A}$ be a σ-algebra on X and $\mathbf{B}_m$ the Borel algebra on R^m. A F-function $f: X \longrightarrow \mathbf{F}(R^m)$ is said to be F-measurable, if $f^{-1}(\widetilde{\mathbf{B}}_m) \subseteq \widetilde{\mathbf{A}}$.

Where $\widetilde{\mathbf{A}} \triangleq \{\mu \in \mathbf{F}(X): \mu^{-1}(\mathbf{B}_1 \cap [0,1]) \subseteq \mathbf{A}\}$ and $\widetilde{\mathbf{B}}_m \triangleq \{\mu \in \mathbf{F}(R^m): \mu^{-1}(\mathbf{B}_1 \cap [0,1]) \subseteq \mathbf{B}_m\}$ are the fuzzy σ-algebras induced from $\mathbf{A}$ and $\mathbf{B}_m$, respectively, in the sense of [5], [6].

Theorem 2.1. A F-function $f: X \longrightarrow \mathbf{F}(R^m)$ is F-measurable iff the set-valued function $f_\alpha: X \longrightarrow \mathbf{P}(R^m)$ is measurable, $\forall \alpha \in (0, 1]$.

Proof. Suppose f is F-measurable, then $f^{-1}(\widetilde{\mathbf{B}}_m) \subseteq \widetilde{\mathbf{A}}$. For $\forall\ C \in \mathbf{C}(R^m)$, we have $C \subseteq \mathbf{B}_m$ and $1_C \in \widetilde{\mathbf{B}}_m$. Thus $f^{-1}(1_C) \in \widetilde{\mathbf{A}}$, and consequently, $[f^{-1}(1_C)]_\alpha = \{x \in X: f_\alpha(x) \cap (1_C)_\alpha \neq \emptyset\} = \{x \in X: f_\alpha(x) \cap C \neq \emptyset\} \in \mathbf{A}$. Hence f_α is measurable, $\forall \alpha \in (0, 1]$.

Conversely, suppose f is measurable. Then $\mu_\alpha \in \mathbf{B}_m$ for $\forall\ \mu \in \widetilde{\mathbf{B}}_m$, and consequently, μ_α is the countable intersections or unious or complements of elements in $\mathbf{C}(R^m)$. Since $\mathbf{A}$ is a σ-algebra on X, we have $[f^{-1}(\mu)]_\alpha = f_\alpha^{-1}(\mu_\alpha) = \{x \in X: f_\alpha(x) \cap \mu_\alpha \neq \emptyset\} \in \mathbf{A}$ and $f^{-1}(\mu) = \cup_{\alpha \in (0,1]} \alpha \cdot [f^{-1}(u)]_\alpha \in \widetilde{\mathbf{A}}$. Hence $f^{-1}(\widetilde{\mathbf{B}}_m) \subseteq \widetilde{\mathbf{A}}$. This means that f is F-measurable. The proof is finished.

Definition 2.2. Let $(X, \mathbf{A}, v)$ be a finite measure space, $f: X \longrightarrow \mathbf{F}(R^m)$ a F-function. For $\forall\ A \in \mathbf{A}$, if $v(A) \neq 0$, the F-integral of f over A is a fuzzy subsets $(F) \int_A f dv \in \mathbf{F}(R^m)$ defined by

$$((F) \int_A f dv)(r) = \sup\{\alpha: r \in (A) \int_A f_\alpha dv, \quad \alpha \in [0, 1]\} \quad \forall\ r \in R^m.$$

Since $r \in (A) \int_A f_0 dv = (A) \int_A R^m dv$ for $\forall\ r \in R^m$, and $(A) \int_A f_\beta dv \subseteq (F) \int_A f_\alpha dv$ for $\forall\ A \in \mathbf{A}$ and $\forall\ 0 \leq \alpha \leq \beta \leq 1$, we know that $(F) \int_A f dv$ is well defined.

Theorem 2.2. $((F) \int_A f dv)_\alpha = \bigcap_{\beta < \alpha} ((A) \int_A f_\beta dv) = (A) \int_A f_\alpha dv$, $\forall\ \alpha \in (0, 1]$

Proof. Straightforward.

Theorem 2.3. Suppose $f: X \longrightarrow \mathbf{F}^*(R^m)$ is F-measurable and f_α is integrably bounded on A for $\forall\ \alpha \in (0, 1]$. Then $(F) \int_A f dv \in \mathbf{F}^*(R^m)$, $A \in \mathbf{A}$.

Proof. By theorem 2.1, we know that $f: X \longrightarrow CoK(R^m) - \{\emptyset\}$ is measurable for $\forall\ \alpha \in (0, 1]$. Therefore it follows from proposition 1.1 and theorem 2.2 that $((F) \int_A f dv)_\alpha = (A) \int_A f_\alpha dv \in CoK(R^m) - \{\emptyset\}$. Hence $(F) \int_A f dv \in \mathbf{F}^*(R^m)$, $A \in \mathbf{A}$.

Definition 2.3. A F-function $f: X \longrightarrow \mathbf{F}(R^m)$ is said to be bounded on A $(A \subseteq X)$, if $U_{x\in A}\text{supp}f(x) = U_{x\in A}\{r \in R^m, (f(x))(r) > 0\}$ is bounded on R^m.

Theorem 2.4. Let $(X, \mathbf{A}, v)$ be a finite measure space and $f: X \longrightarrow \mathbf{F}(R^m)$ a bounded convex F-function. If f_α is closed and A-integrable on A, then the F-valued mapping $F: \mathbf{A} \longrightarrow \mathbf{F}(R^m)$ defined by

$$F(A) = (F) \int_A f dv, \quad A \in \mathbf{A}$$

is a F-number measure on $(X, \mathbf{A})$.

Proof. For $\forall \alpha \in (0, 1]$, let $\Pi_\alpha(A) = (A) \int_A f \, dv$ $(A \in \mathbf{A})$. Then Π_α is a set-valued measure on $(X, \mathbf{A})$[1], and Π_α is bounded convex. We proceed to prove that $\Pi_\alpha(A)$ is compact for $\forall A \in \mathbf{A}$. In fact, suppose $r_n = \int_A g_n dv$ $(g_n \in S(f))$, $n \geqslant 1$, and $r_n \rightarrow r$. Then (cf.[2]) there exists a subsequence $\{g_{nk}\}$ of $\{g_n\}$ and a subsequence $\{h_{nk_i}\}$ of $\{h_{nk}\}$, such that $g_{nk} \xrightarrow{w} g$ and $h_{nk_i} \longrightarrow g$ v-a.e. Where $\{h_{nk}\}$ is the convex combination sequence of $\{g_{nk}\}$. Since f_α is closed convex, we get that $h_{nk} \in f_\alpha$ v-a.e, consequently, $g \in f_\alpha$ v-a.e. and $\int_A g dv = \lim_{k\to\infty} \int_A g_{nk} dv = \lim_{k\to\infty} r_{nk} = r$. Thus $r \in (A)\int_A f_\alpha dv$. This means that $(A) \int_A f_\alpha dv \in \mathbf{K}(R^m)$ is compact. Therefore Π_α is a bounded compact convex set-valued measure.

Moreover, $\Pi_\beta(A) \subseteq \Pi_\alpha(A)$ whenever $0 \leqslant \alpha \leqslant \beta \leqslant 1$. Hence $F(A)(r) = ((F)\int_A f dv)(r) = \sup\{\alpha : r \in \Pi_\alpha(A)\}$ is a F-number measure on $(X, \mathbf{A})$[3].

3. Radon-Nikodym Theorem of F-number Measures

Definition 3.1. Let $(X, \mathbf{A}, v)$ be a finite measure space, $F: \mathbf{A} \longrightarrow \mathbf{F}^*(R^m)$ a fuzzy number measure on $(X, \mathbf{A})$. If there exists a F-function $f: X \longrightarrow \mathbf{F}(R^m)$ such that $F(A) = \int_A f dv$ $(\forall A \in \mathbf{A})$, then we call f the Radon-Nikodym derivative of F with respect to v.

Definition 3.2. Let $(X, \mathbf{A}, v)$ be a finite measure space, $\Pi(F)$ a set-valued (F-number) measure on $(X, \mathbf{A})$. $\Pi(F)$ is said to be absolutely continuous with respect to v, written as $\Pi \ll v$ ($F \ll v$), if $\forall A \in \mathbf{A}$, $v(A) = 0 \Longrightarrow \Pi(A) = \{0\}$ ($F(A) = 1_{\{0\}}$).

Theorem 3.1. Let (X, **A**, v) be a finite measure space and F a F-number measure on (X, **A**). If $F \ll v$, then there exists a F-measurable **R-N** derivative of F.

Proof. For $\forall \alpha \in (0, 1]$, $\Pi_\alpha(A) \triangleq \{r \in R^m: F(A)(r) \geqslant \alpha\}$, $(A \in \mathbf{A})$, is a compact convex set-valued measure[3]. $F \ll v$ means that if $v(A) = 0$ then $F(A) = 1_{\{0\}}$ Thus $\Pi_\alpha(A) = \{r \in R^m: F(A)(r) \geqslant \alpha\} = \{0\}$, i.e. $\Pi_\alpha \ll v$. Hence[2] there exists a F-measurable compact convex set-valued function π_α such that $\Pi_\alpha(A) = (A) \int_A \pi_\alpha dv$ $(A \in \mathbf{A})$ and $\pi_\alpha(x) \subseteq \pi_\beta(x)$ whenever $0 < \alpha \leqslant \beta \leqslant 1$. Let $f = \cup_{\alpha \in (0,1]} \alpha \cdot \pi_\alpha$, then $f: X \longrightarrow \mathbf{F}^*(R^m)$ is F-measurable and $F(A)(r) = \sup \{\alpha : r \in \Pi_\alpha(A)\} = \sup\{\alpha: r \in (A) \int_A \pi_\alpha dv = ((F) \int_A f dv)(r)$, $\forall r \in R^m$. i.e. $F(A) = (F) \int_A f dv$, $\forall A \in \mathbf{A}$. Therefore f is a F-measurable R-N derivative of F. We finish the proof of theorem 3.1.

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