SPLITTING OF FUZZY FIELD EXTENSIONS

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Abstract: We give criteria for a fuzzy field extension to be the fuzzy composite of a fuzzy pure inseparable field extension and a fuzzy separable field extension. We introduce the concept of a distinguished fuzzy intermediate field.

Key words: fuzzy field extension, fuzzy linearly disjoint, fuzzy pure inseparable, fuzzy separable, fuzzy composite, neutral, distinguished.

INTRODUCTION

Let F/K be a field extension of characteristic $p > 0$ and let J denote the maximal purely inseparable intermediate field of F/K. Then F/K is said to **split**, and we write $F = J \otimes_K S$, when F is the field composite JS of J and S over K where S is an intermediate field of F/K which is separable over K. In [5], it is shown that if F/J is separable and J/K has finite exponent, then F/K splits. In [3], it is shown that if F/J has a separating transcendence basis, then F/K splits. An intermediate field S of F/K is called **distinguished** if $F \subseteq K^{p^{-\infty}}S$ and S/K is separable [2]. An example of a field extension is constructed in [1] which does not have a distinguished intermediate field. It is shown in [7] that F/K splits if and only if F/J is separable and F/K has a distinguished intermediate field. We determine in this paper the extent to which these results can be fuzzified, Example 4.2 and Theorems 4.3 and 4.6. Their fuzzification provides basic structure results for inseparable fuzzy field extensions.

A **fuzzy subset** X of F is a function from F into the closed interval [0, 1]. Let X and Y be fuzzy subsets of F. We say that $X \subseteq Y$ if and only if $X(x) \leq Y(x)$ $\forall x \in F$. We say that $X \subset Y$ if and only if $X \subseteq Y$ and $\exists x \in F$ such that $X(x) < Y(x)$. If $\mathcal{I}$ is a collection of fuzzy subsets of F, then we define the fuzzy subsets $I = \bigcap_{X \in \mathcal{I}} X$ and $U = \bigcup_{X \in \mathcal{I}}$ by $\forall x \in F$, $I(x) = \inf\{X(x) \mid X \in \mathcal{I}\}$ and $U(x) = \sup\{X(x) \mid X \in \mathcal{I}\}$. If $c \in F$ and $t \in [0, 1]$, we let c_t denote the fuzzy subset of F defined by $\forall x \in F$, $c_t(x) = t$ if $x = c$ and $c_t(x) = 0$ otherwise. c_t is called a **fuzzy singleton**. If c_t and d_s are fuzzy singletons, then $c_t + d_s = (c + d)_r$ and $c_t d_s = (cd)_r$ where $r = \min\{t, s\}$. A fuzzy subset A of F is called a **fuzzy subfield** of F if and only if $A(1) = 1$, $A(x - y) \geq \min\{A(x), A(y)\}$, and $A(xy^{-1}) \geq \min\{A(x), A(y)\}$, $y \neq 0$. It follows that $\forall x \in F$, $x \neq 0$, $A(0) = A(1) \geq A(x) = A(-x) = A(x^{-1})$. Let A and B be fuzzy subfields of F. If $B \subseteq A$, we write A/B and call A/B a **fuzzy field extension**. We let $\mathcal{F}(A)$ denote the set of all fuzzy subfields C of F such that $C \subseteq A$. If $C \in \mathcal{F}(A)$,

then we let $\mathcal{F}(A/C)$ denote the set of all fuzzy subfields B of F such that $C \subseteq B \subseteq A$. We let Im(A) denote the image of A. $\forall\, t \in [0, 1]$, we let $A_t = \{x \in F \mid A(x) \geq t\}$. We let $A^* = \{x \in F \mid A(x) > 0\}$, the support of A, and $A_\# = \{x \in F \mid A(x) = A(0)\}$. Then A_t ($\forall\, t \in [0, 1]$), A^*, and $A_\#$ are subfields of F. In fact $A_\# = A_1$. If W is a subset of F, we let δ_W denote the characteristic function of W in F. We let N denote the set of positive integers. Needed definitions and results from field theory can be found in [4]. Throughout this paper A denotes a fuzzy subfield of F and $C \in \mathcal{F}(A)$.

1. PRELIMINARY RESULTS

In [10, Definition 3.2], the concept of a neutral fuzzy field extension was introduced. A fuzzy field extension A/B is said to be neutral if and only if $\forall\, x_t \subseteq A$ with $t > 0$, $\exists\, s \in (0, t]$ such that $x_s \subseteq B$. Then A/B is neutral if and only if $A^* = B^*$. The concept of neutrallity plays a fundamental role on the study of fuzzy field extensions.

Proposition 1.1. [8] Let B, D $\in \mathcal{F}(A/C)$. Then B and D are fuzzy linearly disjoint over C [8, Definition 1.15] if and only if B^* and D^* are linearly disjoint over C^* and $B \cap D = C$.

The fuzzy subset A^P of F, defined by $A^P(x) = A(z)$ if $x = z^P$ for some $z \in F$ and $A^P(x) = 0$ otherwise, is such that $A^P \in \mathcal{F}(A)$ by [8, Theorem 2.9].

Proposition 1.2. [8] Let B $\in \mathcal{F}(A)$. Then A/B is fuzzy separable [8, Definition 3.1] if and only if A^*/B^* is separable and $B \cap A^P = B^P$.

Proposition 1.3. Let B, D $\in \mathcal{F}(A/C)$. Suppose that B^* and D^* are linearly disjoint over C^*. Then

(i) $(B \cap D)/C$ is neutral;

(ii) B and D are fuzzy linearly disjoint over $B \cap D$;

In [9, Definition 1.1] and [10, Definition 3.1] two different fuzzifications of the concept of an algebraic field extension were introduced and compared. This then led of course to two different fuzzifications of the concepts of a purely inseparable field extension and a separable algebraic field extension [9, 10].

The following example shows that a neutral fuzzy field extension need not be fuzzy separable.

Example 1.4. Let F = P(x) and K = P(x^P) where P is a perfect field of characteristic p > 0 and x is transcendental over P. Let $A = \delta_F$. Define the fuzzy subset B of F by B(z) = 1 if z $\in$ K and B(z) = 1/2 otherwise. Then B $\in \mathcal{F}(A)$. Now $(B \cap A^P)(x^P) = \min\{B(x^P), A^P(x^P)\} = \min\{B(x^P), A(x)\} = 1$ and $B^P(x^P) = B(x) = 1/2$. Thus $B \cap A^P \neq B^P$. Hence A/B is not fuzzy separable. However, A/B is neutral and thus fuzzy separable algebraical, [10, Definition 4.2]. Now $(B \cap A^P)^* = $

$B^* \cap (A^P)^* = B^* \cap (A^*)^P = F^P = (B^*)^P = (B^P)^*$ by [8, Proposition 2.12]. Thus $(B \cap A^P)/B^P$ is neutral.

Example 1.4 also shows that a fuzzy separable algebraical field extension need not be fuzzy separable according to the current definition of fuzzy separable. This illustrates another difference between the concepts of fuzzy separable algebraic [9, Definition 1.10] and fuzzy separable algebraical since a fuzzy separable algebraic field extension is necessarily fuzzy separable, [8, Proposition 3.9].

Proposition 1.5. Let $B \in \mathcal{F}(A)$. Suppose that A/B is neutral. Then A/B is fuzzy separable if and only if $B \cap A^P = B^P$.

Definition 1.6. Let $B, D \in \mathcal{F}(A)$. Define the fuzzy subset BD of F by $\forall x \in F$, $(BD)(x) = \sup\{\min\{\min\{B(y_i), D(z_i)\} \mid i = 1, ..., n\} \mid x = \sum_{i=1}^{n} y_i z_i, n \in N\}$. BD is called the **fuzzy composite** of B and D.

By [8, Theorem 1.5], BD is a fuzzy subring of F. By the notation $A = B \otimes D$, we mean that $A = BD$ and that B, D are fuzzy linearly disjoint (over $B \cap D$). By the notation $A^* = B^* \otimes D^*$, we mean the tensor product of B^* and D^* over $B^* \cap D^*$.

Proposition 1.7. Let $B, D \in \mathcal{F}(A)$. Then $A = B \otimes D$ if and only if $A^* = B^* \otimes D^*$ and $A = BD$.

Proposition 1.8. Let $B, D \in \mathcal{F}(A)$. Then

(i) $A = BD$ if and only if $A_t = (BD)_t \forall t \in [0, 1]$;

(ii) $(B \cap D)_t = B_t \cap D_t \forall t \in [0, 1]$;

(iii) $B_t D_t \subseteq (BD)_t$ and if B and D have the sup property, then $B_t D_t = (BD)_t \forall t \in [0, 1]$.

Proposition 1.9. Let $B, D \in \mathcal{F}(A)$. Let $K = B^* \cap D^*$. Suppose that $B_\# \cap D_\# \supseteq K$. Then B and D are fuzzy linearly disjoint (over $B \cap D$) if and only if B_t and D_t are linearly disjoint over $K \forall t \in (0, 1]$.

Proposition 1.10. Let $B, D \in \mathcal{F}(A)$. Let $K = B^* \cap D^*$. Suppose that $B_\# \cap D_\# \supseteq K$.

(i) Suppose that B and D have the sup property. If $A = B \otimes D$, then $A_t = B_t \otimes_K D_t \forall t \in (0, 1]$.

(ii) if $A_t = B_t \otimes_K D_t \forall t \in (0, 1]$, then $A = B \otimes D$.

2. NEUTRALLY CLOSED SUBFIELDS

Let $B \in \mathcal{F}(A)$. We say that B is **neutrally closed** in A if and only if whenever $x_t \subseteq A$ and $x_s \subseteq B$ with $t \geq s = B(x) > 0$, then $t = s$. Clearly B is neutrally closed in A if and only if $A = B$ on

B*.

Proposition 2.1. Let B, D $\in$ $\mathcal{F}$(A/C). Suppose that C is neutrally closed in A.

(i) A/C is fuzzy separable if and only if A^*/C^* is separable.

(ii) B and D are fuzzy linearly disjoint over C if and only if B^* and D^* are linearly disjoint over C^*.

Let B $\in$ $\mathcal{F}$(A). It was shown in [10, Example 4.9] that B does not necessarily have a fuzzy algebraic closure in A.

Proposition 2.2. Let B $\in$ $\mathcal{F}$(A). Suppose that inf$\{A(b) \mid b \in B^*\}$ $\geq$ sup$\{A(x) \mid x \notin B^*\}$. If B is neutrally closed in A, then B has a fuzzy algebraic closure in A.

In Proposition 2.2, B^C is neutrally closed in A since $B^C = A$ on B^{C*}.

Proposition 2.3. Let B $\in$ $\mathcal{F}$(A). Suppose that B is neutrally closed in A. If A/B is fuzzy pure inseparable [10, Definition 4.1], then A/B is fuzzy purely inseparable [9, Definition 1.6].

If A/B is fuzzy purely inseparable, then clearly A/B is fuzzy pure inseparable. Thus Proposition 2.3 gives a condition under which A/B is fuzzy purely inseparable if and only if A/B is fuzzy pure inseparable, namely that B is neutrally closed in A. In fact, the assumption that B be neutrally closed in A yields the result that if $x_t \subseteq A$, $t > 0$, is fuzzy algebraical over B with respect to

the polynomial $p(y) = y^{p^e} + b$ over B^* [10, Definition 3.7], then x_t' is also fuzzy algebraic over B with respect to $p(y)$ over B^*.

Example 2.4. Let F = P(x) where P is a perfect field of characteristic p > 0 and x is transcendental over P. Let A = δ_F. Define the fuzzy subset B of F by B(y) = 1 if y $\in$ $P(x^p)$ and B(y) = 1/2 otherwise. Then B $\in$ $\mathcal{F}$(A). Now $y_t + (-y)_{1/2} = 0_{1/2}$ and $(y_s)^p + (-y_s)^p = 0_s$ $\forall$ y $\in$ F and $\forall$ t, s $\in$ [0, 1], t $\geq$ 1/2. Thus A/B is both fuzzy pure inseparable and fuzzy purely inseparable, yet B is not neutrally closed in A.

3. DISTINGUISHED FUZZY SUBFIELDS

Let B $\in$ $\mathcal{F}$(A/C). We let $C^{*\infty}$ denote $C^{*p^{-\infty}}$ where $C^{*p^{-\infty}} = \{c^{p^{-n}} \mid c \in C^*, n \in N \cup \{0\}\}$. We sometimes think of B as being extended to $C^{*\infty}F$ such that B(x) = 0 $\forall$ x $\in$ $C^{*\infty}F - F$. For fuzzy subfields $\delta_{C^{*\infty}}$ and D of $C^{*\infty}F$, $\delta_{C^{*\infty}}D$ denotes the fuzzy composite of $\delta_{C^{*\infty}}$ and D.

If B $\in$ $\mathcal{F}$(A) and X is a fuzzy subset of F such that X $\subseteq$ A, then B(X) denotes the intersection of all C $\in$ $\mathcal{F}$(A) such that B $\cup$ X $\subseteq$ C and B[X] denotes the intersection of all fuzzy subrings R of F such that B $\cup$ X $\subseteq$ R $\subseteq$ A.

Proposition 3.1. Let B, D $\in$ $\mathcal{F}(A/C)$. If B/C is fuzzy algebraical , then BD is a fuzzy subfield of F.

Let B $\in$ $\mathcal{F}(A/C)$. Then B is called the **unique maximal fuzzy pure inseparable intermediate field** of A/C if and only if B/C is fuzzy pure inseparable and if x_t $\subseteq$ A with t $>$ 0, is fuzzy pure inseparable over C, then x_t $\subseteq$ B. That B exists follows from [10, Proposition 4.8].

Definition 3.2. A/C is said to have a **distinguished fuzzy intermediate field** D if and only if $\exists$ D $\in$ $\mathcal{F}(A/C)$ such that D/C is fuzzy separable and A $\subseteq$ $\delta_{C^*\infty}D$.

Proposition 3.3. Let D $\in$ $\mathcal{F}(A/C)$. Suppose that D^* is distinguished in A^*/C^*. Then $\delta_{C^*\infty}A/\delta_{C^*\infty}D$ is neutral.

Proposition 3.4. Let D $\in$ $\mathcal{F}(A/C)$. Suppose that D^* is distinguished in A^*/C^*. If D is neutrally closed in A and sup$\{A(c) \mid c \in A^* - D^*\}$ $\leq$ inf$\{A(d) \mid d \in D^*\}$, then $\delta_{C^*\infty}D = \delta_{C^*\infty}A$.

Proposition 3.5. Let B be the unique maximal fuzzy pure inseparable intermediate field of A/C. Then B is neutrally closed in A if and only if $\delta_{C^*\infty} \cap A = B$.

4. SPLITTING

Definition 4.1. A/C is said to split if and only if $\exists$ B, D $\in$ $\mathcal{F}(A/C)$ such that B/C is fuzzy pure inseparable, D/C is fuzzy separable, and A = B $\otimes_C$ D, i. e., A = B $\otimes$ D and B $\cap$ D = C.

The following example illustrates complications which arise in fuzzifying structure results from the crisp case. A/C does not split even though $\exists$ B $\in$ $\mathcal{F}(A/C)$ such that A/B has a fuzzy separating transcendence basis [10, Definition 4.6] and B^p $\subseteq$ C. Also A/C does not have a distinguished fuzzy intermediate field.

Example 4.2.

Theorem 4.3. Let B $\in$ $\mathcal{F}(A/C)$.

(i) If A/B has a fuzzy separating transcendence basis and B/C is fuzzy pure inseparable, then $\exists$ D $\in$ $\mathcal{F}(A/C)$ such that D/C has a fuzzy separating transcendence basis, BD = B $\otimes$ D and A/(B $\otimes$ D) is neutral.

(ii) If A/B is fuzzy separable and B/C is fuzzy pure inseparable of bounded exponent, then $\exists$ D $\in$ $\mathcal{F}(A/C)$ such that BD = B $\otimes$ D and A/(B $\otimes$ D) is neutral.

In Theorem 4.3(ii), D/C is not necessarily fuzzy separable: Let B/C be neutral, but not fuzzy separable as in Example 1.4. Let A = B. Then A/B is fuzzy separable and B/C is fuzzy pure inseparable of bounded exponent. Now D = A, but D/C is not fuzzy separable.

Let B be the unique maximal fuzzy pure inseparable intermediate field of A/C. If A/B is fuzzy separable and $\exists$ D $\in$ $\mathcal{F}(A/C)$ which is distinguished for A/C, then A/BD is neutral: A $\subseteq$ $\delta_{C^*\infty}D$ and so $A^* \subseteq (\delta_{C^*\infty}D)^* = C^{*\infty}D^*$. Thus D^* is distinguished for A^*/C^*. Since A^*/B^* is separable and B^*/C^* is purely inseparable, $A^* = B^* \otimes_{C^*} D^* = (BD)^*$ by [7, Theorem 1, p. 606].

Theorem 4.4. [8, Theorem 1.20] Let E, D $\in$ $\mathcal{F}(A/C)$ and B $\in$ $\mathcal{F}(E/C)$. Suppose that E and D are fuzzy linearly disjoint over C and BD is a fuzzy subfield of F. Then (i) implies (ii) implies (iii) where

(i) $C^* \subseteq B_{\#}$;

(ii) B = BD on B^*;

(iii) E $\cap$ BD = B.

Theorem 4.5. Let B be the unique maximal fuzzy pure inseparable intermediate field of A/C. Suppose that $B^* \subseteq B_{\#}$ and that C is neutrally closed in B. Then A/C splits if and only if A/B is fuzzy separable and $\exists$ D $\in$ $\mathcal{F}(A/C)$ such that D is distinguished for A/C.

REFERENCES

1. J. Deveney, A counterexample concerning inseparable field extensions, Proc. Amer. Math. Soc. 55 (1976), 33 - 34.

2. J. Dieudonne, Sur les extensions transcendantes separables, Summa Brasil. Math. 2 (1947), 1 - 20,

3. N. Heerema and D. Tucker, Modular field extensions, Proc. Amer. Math. Soc. 53 (1975), 301 - 306.

4. N. Jacobson, Lectures in abstract algebra, Vol. III: Theory of fields and Galois theory. Princeton 1964.

5. H. Kreimer and N. Heerema, Modularity vs. separability for field extensions, Canad. J. Math. 27 (1975), 1176 - 1182.

6. D. Malik and J. Mordeson, Fuzzy subfields, Fuzzy Sets and Systems (to appear).

7. J. Mordeson, Splitting of field extensions, Archiv Der Mathematik, Vol. XXVI (1975), 606 - 610.

8. J. Mordeson, Fuzzy field extensions, preprint.

9. J. Mordeson, Fuzzy algebraic field extensions, Fuzzy Sets and Systems (to appear).

10. J. Mordeson, Fuzzy transcendental field extensions, preprint.

11. S. Nanda, Fuzzy fields and linear spaces, Fuzzy Sets and Systems 19 (1986), 89 - 94.

12. M. Osman, On t-fuzzy subfields and t-fuzzy vector spaces, Fuzzy Sets and Systems 33 (1989), 111 - 117.

13. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512 - 517.

14. L. Zadeh, Fuzzy Sets, Inform. and Control 8 (1965), 338 - 353.