FUZZY POLYNOMIAL RINGS

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In this paper we intruduce a concept of a fuzzy graded ring by natural numbers, a fuzzy polynomial ring and prove some related theorems.

1. INTRODUCTION

Zadeh [6] introduced the concept of fuzzy set. Rosenfeld [5] applied this concept to the theory of groups and groupoids. In [2,3], Liu introduced and developed basic results concerning the notions of fuzzy subrings and fuzzy ideals of a ring. Malik and Mordeson [4] introduced the concept of fuzzy direct sums of fuzzy subrings and fuzzy ideals of a ring. The purpose of this paper is to introduce the concept of a fuzzy graded ring and a fuzzy polynomial ring.

We let L denoted a completely distributive lattice with greatest element 1 and least element 0. If X is a nonempty set, an L-fuzzy set in X, or merely a fuzzy set in X, is a map $A:X\longrightarrow L$, and $\mathscr{C}(X)$ will denote the set of all fuzzy sets in X. We say that L has the (finite) intersection property if the infimum of every (finite) set of nonzero element of L is nonzero.

Throughout this paper X is a commutative ring with identity. Let R be a fuzzy subset of X such that $R(0)=1$. If $R(x)\wedge R(y)\leq R(x-y)$ for all $x,y\in X$, we call R a fuzzy group in X and if a fuzzy group in X satisfies $R(x)\wedge R(y)\leq R(xy)$ we call R a fuzzy ring in X.

2. FUZZY DIRECT SUMS

Let I be a nonempty set. Let $x, x_i\in X$ where $i\in I$. By $x=\sum_{i\in I} x_i$ we mean that all but finitely many of the x_i are zero.

Definition 2.1 [4]. Let $\{R_i \mid i\in I\}$ be a collection of fuzzy subsets of X. We define the fuzzy subset $\sum_{i\in I} R_i$ of X by $(\sum_{i\in I} R_i)(x)=\sup\{\inf\{R_i(x_i)\} \mid i\in I \mid x=\sum_{i\in I} x_i\}$ for all $x\in X$.

It is proved that if $\{R_i \mid i\in I\}$ is a collection of fuzzy groups, rings (fuzzy ideals) in X, then $\sum_{i\in I} R_i$ is a fuzzy group, ring (fuzzy ideal) in X respectively and $R_i\leq \sum_{i\in I} R_i$. [4].

Definition 2.2. Let R, R_i where $i\in I$ be fuzzy groups (rings) in X. Then R is said to be the fuzzy (weak) direct sum of $\{R_i \mid i\in I\}$ if $R=\sum_{i\in I} R_i$ and $R_j\cap\sum_{i\neq j} R_i=E$ where

$$E(x)=\begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases}$$

. In this case we write $R=\bigoplus_{i\in I} R_i$

3. FUZZY GRADED SUBRINGS

Definition 3.1. Let X be a commutive ring (with identity). The fuzzy ring R in X is said to be graded by natural numbers if there exists fuzzy groups R_i, $i=0,1,2,\dots$ in X such that

(i) $R=\bigoplus_{i\in I} R_i$ and

(ii) $R_iR_j\subseteq R_{i+j}$ for all $i,j=0,1,2,\dots$, where for all $x\in X$

$$R_iR_j(x)=\sup\left\{\inf\{R_i(y_k)\wedge R_j(z_k)\mid k=1,2,\dots,n\}\mid x=\sum_{k=1}^{n} y_kz_k,\quad n\in\mathbb{N}\right\}.$$

From $R_oR_o\subset R_o$ it follows that R_o is a fuzzy subring of X.

Example 3.2. Let R be any fuzzy subring of X. Define $R_o=R$ and $R_i=E$ for $i=1,2,\dots$. Then R has a trivial gradation $\{R_i\}_{i=o}^{\infty}$.

Example 3.3. Let $\mathcal{B}$ be the totality of infinite sequences $(a_o,a_1,a_2,\dots)$, $a_i\in X$ that have only a finite number of nonzero terms a_i. Define

$$(a_o,a_1,\dots)+(b_o,b_1,\dots)=(a_o+b_o,a_1+b_1,\dots) \text{ and}$$
$$(a_o,a_1,\dots).(b_o,b_1,\dots)=(p_1,p_2,\dots) \text{ when}$$

$p_i=\sum_{j+k=i} a_jb_k$. Then $(\mathcal{B},+,.)$ is a ring [1]. Let A be a fuzzy subring of X. Define R and R_i, $i=0,1,\dots$ on $\mathcal{B}$ as follows:

$$R(a_o,a_1,\dots)=\inf\{A(a_i)\mid i=0,1,\dots\} \text{ and}$$

$$R_i(a_o,a_1,\dots)=\begin{cases}A(a_i) & \text{if } a_j=0 \text{ for all } j\neq i\\ 0 & \text{if } a_j\neq 0 \text{ for some } j\neq i\end{cases}$$

It is clear that R is a fuzzy subring of $\mathcal{B}$ and R_i, $i=0,1,\dots$ is a fuzzy subgroup of $\mathcal{B}$. It can be readily verified that

$$R=\bigoplus_{i=0}^{\infty} R_i \text{ amd } R_iR_j\subseteq R_{i+j} \text{ for all } i,j=0,1,\ldots .$$

The above example motivates the following definition of a fuzzy polynomial subring.

Definition 3.3. Let $X[u]$ be the polynomial ring of one indeterminate u on X. A <u>fuzzy polynomial subring</u> R of $X[u]$ is a fuzzy subring of $X[u]$ such that for all $p(u)\in X[u]$, $R(p(u))=R(a_0+a_1u+\ldots+a_nu^n)=\inf\{A(a_i)\mid i=0,\ldots,n\}$ for some fuzzy subring A of X.

Recall that if R is a fuzzy subring of X, then $R_*=\{x\in X\mid R(x)=1\}$ is a fuzzy subring of R and if L has the finite intersection property, then $R^*=\{x\in X\mid R(x)>0\}$ is a subring of X. Based on these facts we prove:

Proposition 3.4. Let R be a fuzzy graded subring of X. If L has the finite intersection property, then R^* is a graded subring of X.

Definition 3.5. Let R be a graded fuzzy subring of X and $x_\lambda\in R$ $(R(x)=\lambda\neq 0)$. The value $R(x_i)=\lambda_i$ is called the <u>homogeneous component</u> of x_λ of degree i.

Theorem 3.6. Let $X=\bigoplus_{i=0}^{\infty} X_i$ be a commutative graded ring with identity. Let L satisfy the finite intersection property and R be an L-fuzzy subring of X such that $R(\sum_{i=0}^{\infty} x_i)=\bigwedge_{i=0}^{\infty} R(x_i)>0$. Then R is a fuzzy graded subring of X.

4. FUZZY GRADED MODULES

Let R be a fuzzy subring of X and Y be an X-module the fuzzy subset M of Y is called a fuzzy R-module if for all y_1, y_2, y in Y and x in X

(i) $M(y_1-y_2)\geq M(y_1)\wedge M(y_2)$, and

(ii) $M(xy)\geq R(x)\wedge M(y)$

Definition 4.1. Let R be a fuzzy graded subring of X. A fuzzy R-module M is called fuzzy graded R-module if M can be expressed as a direct sum of fuzzy subgroups $\{M_i\}$, i.e., $M=\sum_{i=0}^{\infty} M_i$, such that $R_iM_j\leq M_{i+j}$, $i,j=0,1,\ldots$ as fuzzy subsets of Y.

Theorem 4.2. Let R be a fuzzy graded subring of X, M a fuzzy graded R-module. If L satisfies the finite intersection property, then M^* is a graded R^*-module.

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