Uniform HX Group

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Abstract

In the paper[1] the upgrade of the structure of groups has been considered, in which the concept of HX group has been raised. In this paper we give the definition of uniform HX group and discuss its structure.

Key words: HX group, uniform HX group.

1. Introduction

The so-called HX group is a group which is formed on the power set P(G) of a group G. Let $(G,\cdot)$ be a group, by using multivariate extension principle[2], the operation of the group G

$$G\times G\longrightarrow G,\ (a,b)\longmapsto a\cdot b\triangleq ab$$

can be extended to P(G). For any $A,B\in P(G)$

$$AB=\{ab\mid a\in A,\ b\in B\} \qquad (1)$$

Let $\mathfrak{G}\subseteq P_0(G)\triangleq P(G)-\{\phi\}$, $\mathfrak{G}\neq\phi$. If $\mathfrak{G}$ forms a group for the operation(1) then, $\mathfrak{G}$ is called a HX group on G, which its unit element is denoted by E and for any $A\in\mathfrak{G}$, the inverve element of A is denoted by A^{-1}.

We point out that a quotient group on G must be a HX group on G and its unit element be a normal subgroup of G.

2. Uniform HX group

If the operation of inverse element in G is upgraded to P(G) by means of extension principle, then we can seek an inverse set in P(G).

<u>Definition 2.1</u> Let G be a group. For any $A \in P(G)$

$$A^{\ominus} \triangleq \{x^{-1} \mid x \in A\}$$

is called an inverse set of A.

So, we have an inverse element A^{-1} and an inverse set $A^{\ominus}$ for a arbitary element A of a HX group. It is well worth attention that there is difference between $A^{\ominus}$ and A^{-1}. Generally, the inverse set of A is not uniform with the inverse element of A in a HX group $\mathcal{G}$. For example, let G be the additive group of real numbers, take $E=(0,+\infty)$, then

$$\mathcal{G} \triangleq \{x+E \mid x \in G\}$$

is a HX group on G, and its unit element is just E. $\mathcal{G}$ is a additive group too. Here the inverse element of A and the inverse set of A are respectively called the negative element of A and the negative set of A, write $-A$ and $\ominus A$. Obviously, for E, $-E=E=(0,+\infty)$, $\ominus E=(-\infty,0)$. So, $-E \neq \ominus E$.

It is well worth research that under what condition $A^{\ominus}=A^{-1}$ holds. In this section we will discuss this kind of group in which the inverse set is uniform with the inverse element.

<u>Definition 2.2</u> A HX group $\mathcal{G}$ is called an uniform HX group if

for any $A \in \mathfrak{G}$, $A^{-1} = A^{\textcircled{-1}}$ holds.

<u>**Theorem 2.1**</u> Let H be a normal subgroup of G, then the quotient group G/H is an uniform HX group on G.

<u>**Proof.**</u> $G/H = \{aH \mid a \in G\}$. For any $aH \in G/H$, we have $(aH)^{-1} = a^{-1}H$. From definition 2.1,

$$(aH)^{\textcircled{-1}} = \{(ah)^{-1} \mid h \in H\} = \{h^{-1}a^{-1} \mid h \in H\} = Ha^{-1} = a^{-1}H.$$

So, we have $(aH)^{\textcircled{-1}} = (aH)^{-1}$.

Therefore, from definition 2.2, G/H is an uniform HX group on G.

<u>**Theorem 2.2**</u> Let $\mathfrak{G}$ be a HX group on G, then, $\mathfrak{G}$ is uniform iff its unit element E is a subgroup of G.

<u>**Proof.**</u> (1). Let $\mathfrak{G}$ be an uniform HX group on G. For the unit element E of $\mathfrak{G}$, we have $E^{-1} = E^{\textcircled{-1}}$. For any $a, b \in E$, $ab^{-1} \in EE^{\textcircled{-1}} = EE^{-1} = E$. Therefore, E is a subgroup of G.

(2). Let the unit element E of $\mathfrak{G}$ be a subgroup of G. First of all, we prove that for any $A \in \mathfrak{G}$, any $a \in A$, $A = aE = Ea$ holds. Obviously, we have $aE \subseteq AE = A$. If $aE \subsetneq A$, then there exists $b \in A$ such that $b \notin aE$. We have $b^{-1}a \notin E$. For $d \in A^{-1}$, $db, da \in A^{-1}A = E$. Since E is a subgroup of G, $(db)^{-1}(da) \in EE = E$, i.e. $b^{-1}a \in E$. This is in contradication with $b^{-1}a \notin E$. So, $A = aE$ holds. Similarly we have $A = Ea$.

Next, we prove $A^{\textcircled{-1}} = A^{-1}$. For any $a \in A^{-1}$, noting $A^{-1}A = E$ and $e \in E$, where e is the unit element of G. Then there exists $b \in A^{-1}$, $b' \in A$ such that $bb' = e$. Therefore $b^{-1} = b' \in A$. Since $A^{-1} = bE$, there exists $c \in E$ such that $a = bc$. We have

$$a^{-1}=c^{-1}b^{-1}\in Eb^{-1}=A.$$

From definition 2.1, $a\in A^{(-1)}$, So $A^{-1}\subseteq A^{(-1)}$ holds. Conversely, for any $a\in A^{(-1)}$, we have $a^{-1}\in A$. Similarly we can prove $a\in A^{-1}$. So, $A^{(-1)}\subseteq A^{-1}$ holds.

Therefore $A^{-1}=A^{(-1)}$ holds. From definition 2.2, $\mathfrak{G}$ is an uniform HX group on G.

<u>**Theorem 2.3**</u> Let $\mathfrak{G}$ be a HX group on G. If the step of every element in the unit element E of $\mathfrak{G}$ is finite, then $\mathfrak{G}$ is an uniform HX group on G.

<u>**Proof.**</u> Because E is the unit element of $\mathfrak{G}$, we have $EE=E$. Obviously, $E^n=E$ $(n=1,2,\ldots)$ holds. Let the step of a be m, for any $a\in E$. So, $a^m\in E^m=E$, i.e. $e\in E$, and $a^{-1}=a^{m-1}\in E^{m-1}=E$. Therefore E is a subgroup of G. From theorem 2.2, $\mathfrak{G}$ is an uniform HX group on G.

<u>**Corallary 2.1**</u> If the unit element E of HX group $\mathfrak{G}$ is finite, then $\mathfrak{G}$ is an uniform HX group on G.

<u>**Corallary 2.2**</u> If G is a finite group then a HX group $\mathfrak{G}$ on G must be uniform.

<u>**Thenrem 2.4**</u> Let $\mathfrak{G}$ be an uniform HX group on G, then

$$G^*\triangleq\cup\{A\mid A\in\mathfrak{G}\}$$

is a subgroup of G.

<u>**Proof.**</u> First of all, since $\mathfrak{G}$ is a HX group on G we have $G^*\neq\phi$. For any $a,b\in G^*=\cup\{A\mid A\in\mathfrak{G}\}$, there exist $A,B\in\mathfrak{G}$ such that $a\in A$, $b\in B$. Because $\mathfrak{G}$ is a group we have $AB\in\mathfrak{G}$. So $ab\in AB\subseteq G^*$. In addition for any $a\in G^*$ where $a\in A$, $A\in\mathfrak{G}$, we have $A^{(-1)}=A^{-1}$ because $\mathfrak{G}$ is an uniform HX group. So, $a^{-1}\in A^{(-1)}=A^{-1}\subseteq G^*$. Therefore G^* is a subgroup of G.

Theorem 2.5 Let $\mathfrak{G}$ be an uniform HX group on G, and E be the unit element of $\mathfrak{G}$, then E is a normal subgroup of G^* and $\mathfrak{G}=G^*/E$.

Proof. Since $\mathfrak{G}$ is an uniform HX group and E is the unit element of $\mathfrak{G}$, E is a subgroup of G. In addition since $E\subseteq G^*$ and G^* is a subgroup of G, E is also a subgroup of G^*. For any $a\in G^*$, there exists $A\in\mathfrak{G}$ such that $a\in A$. From the proof of theorem 2.2 we have A=aE=Ea. Hence, E is a normal subgroup of G^*.

Now, let us prove that $\mathfrak{G}=G^*/E=\{aE \mid a\in G^*\}$. For any $A\in\mathfrak{G}$, take $a\in A\subseteq G^*$, we have $A=aE\in G^*/E$. On the other hand, for any $aE\in G^*/E$, where $a\in G^*$, there exists $A\in\mathfrak{G}$ such that $a\in A$. So, $aE=A\in\mathfrak{G}$. Therefore, $\mathfrak{G}=G^*/E$.

The theorem 2.5 shows an uniform HX group is just some quotient group.

REFERENCES

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[2] Luo Chengzhong, Theory of fuzzy sets, Beijing Normal University (1989).