ON THE ANTI FUZZY SUBGROUPS UNDER t-NORMS

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1. Introduction and Preliminaries

Since the concept of fuzzy subgroups was introduced by Rosenfeld[1], it has been studying by several authors in [2-11]. Recently, Biswas has proposed the concept of anti fuzzy subgroups[8]. In this paper, we will generalize this concept to that of anti fuzzy subgroups under t-norms, and investigate some of their properties. We will also study the problems of the anti products and isomorphisms of anti fuzzy subgroups under t-norms.

Throughout this paper, let G be a group, $I = [0, 1]$. We will denote a t-norm by T and refer for its properties to [3,4,6,9,12].

Definition 1.1. Let T_1 and T_2 be t-norms and $f: I \longrightarrow I$ an order-preserving bijection. We say that T_2 is the conjugate of T_1, written as $\overline{T}_1$, if

$$\overline{T}_1(a,b) = T_2(a,b) = 1 - T_1(1-a,1-b) \quad \forall\ a,b \in I$$

and that T_2 dominates T_1, written as $T_2 \gg T_1$ or $T_1 \ll T_2$, if

$$T_2(T_1(a,b),T_1(c,d)) \geqslant T_1(T_2(a,c),T_2(b,d)) \quad \forall\ a,b,c,d \in I$$

and that T_2 is generated by T_1 via f, if

$$T_2(a,b) = f^{-1}(T_1(f(a),f(b))) \quad \forall\ a,b \in I$$

Definition 1.2. Let X be an ordinary set. By a fuzzy subsets u of X, we mean a function $u: X \longrightarrow I$ with $u(x)$ as the grade of membership for $\forall\ x \in X$.

Definition 1.3.[3] A fuzzy subgroug of G under a t-norm T (called T-fuzzy subgroup of G, for short) is a fuzzy subset u of G satisfying

(1) $u(xy) \geqslant T(u(x),u(y)) \quad \forall\ x,y \in G$

(2) $u(x^{-1}) = u(x)$ where x^{-1} is the inverse of x, $\forall\ x \in G$

(3) $u(e) = 1$ where e is the identity of G.

Definition 1.4. [8] A fuzzy subset u of G is called an anti fuzzy subgroup of G, if $u(xy) \leq \max(u(x),u(y))$ and $u(x^{-1}) \leq u(x)$ for $\forall\, x,y \in G$.

2. Anti Fuzzy Subgroups Under t-Norms

Definition 2.1. A fuzzy subset u of G is called an anti fuzzy subgroups of G under a t-norm T (called anti T-fuzzy subgroup of G, for short), if

(1) $u(xy) \leq \overline{T}(u(x),u(y))$ where $\overline{T}$ is the conjugate of T, $\forall\, x,y \in G$

(2) $u(x^{-1}) = u(x)$ where x^{-1} is the inverse of x, $\forall\, x \in G$

(3) $u(e) = 0$ where e is the identity of G.

Based on the definitions above, we have the following properties of anti T-fuzzy subgroups omitting the proofs:

Proposition 2.1. A fuzzy subset u of G is an anti T-fuzzy subgroup of G iff its complement u^c, defined by $u^c(x) = 1 - u(x)$ $\forall\, x \in G$, is a T-fuzzy subgroup of G.

Proposition 2.2. A fuzzy subset u of G is an anti T-fuzzy subgroup of G iff $u(e) = 0$ and $u(xy^{-1}) \leq T(u(x),u(y))$ $\forall\, x,y \in G$.

Proposition 2.3. Let T be a t-norm satisfying $T(a,b) < 1$ for $\forall\, a,b \in [0, 1)$. If u is an anti T-fuzzy subgroup of G, then $L(u) = \{x \in G: u(x) < 1\}$ is a subgroup of G.

Definition 2.2. Let X and Y be ordinary sets and $h: X \longrightarrow Y$ be a mapping. If u is a fuzzy subset of X, then the fuzzy subset h(u) of Y defined by

$$[h(u)](y) = \begin{cases} \inf\limits_{x \in h^{-1}(y)} u(x) & \text{if } y \in h(X) \\ 0 & \text{otherwise} \end{cases}$$

is called the image of u under h.

If u is a fuzzy subset of Y, then the fuzzy subset $h^{-1}(u)$ of X defined by

$$[h^{-1}(u)](x) = u(h(x)) \qquad \forall\, x \in X$$

is called the preimage of u under h.

Proposition 2.4. Let h be a homomorphism of group G into group G'.

(1) If u is an anti T-fuzzy subgroup of G'; then $h^{-1}(u)$, the preimage of u under h, is an anti T-fuzzy subgroup of G.

(2) If T is a continuous t-norm and u is an anti T-fuzzy subgroup of G; then h(u), the image of u under h, is an anti T-fuzzy subgroup of G'.

3. Anti Products Under t-Norms of Anti T-Fuzzy Subgroups

Definition 3.1. Let u and v be fuzzy subsets of G. The anti product of u and v under a t-norm T (called anti T-product of u and v, for short), written as $[u.v]_{\bar{T}}$, is a fuzzy subset of G defined by

$$[u.v]_T(x) = T(u(x),v(x)) \quad \forall\, x \in G$$

Based on the properties of anti T-fuzzy subgroups, we have the following properties of anti T-products and omit the proofs:

Proposition 3.1. Let T_1 and T_2 be t-norms and $T_2 \gg T_1$. If u and v are anti T_1-fuzzy subgroups of G; then $[u.v]_{T_2}$, the anti T_2-product of u and v, is also an anti T_1-fuzzy subgroup of G.

Proposition 3.2. Let T_1 and T_2 be t-norms and $T_2 \gg T_1$, u and v be anti T_1-fuzzy subgroups of G. If h is a homomorphism of G into a group G', then

(1) The preimage of $[u.v]_{T_2}$ under h, $h^{-1}([u.v]_{T_2})$, is also an anti T_1-fuzzy subgroup of G.

(2) $h^{-1}([u.v]_{T_2}) = [h^{-1}(u).h^{-1}(v)]_{T_2}$.

Proposition 3.3. Let T_1 be a continuous t-norm and the t-norm T_2 dominates T_1, u and v be anti T_1-fuzzy subgroups of G'. If h is a homomorphism of G into G', then

(1) The image of $[u.v]_{T_2}$ under h, $h([u.v]_{T_2})$, is also an anti T_1-fuzzy subgroup of G'.

(2) $h([u.v]_{T_2}) \supseteq [h(u),h(v)]_{T_2}$.

4. Isomorphisms of Anti T-Fuzzy Subgroups

Let u_i be an anti T_i-fuzzy subgroup of G_i, M_i denote the subgroup of G_i generated by the subset $L(u_i) = \{x \in G_i : u_i(x) < 1\}$. $U_i = \{u_i(x) : x \in L(u_i)\} \subseteq I$ and $\overline{T}_i(U_i) = \{\overline{T}_i(a,b) : a,b \in U_i\} \subseteq I$. $i=1,2$.

Definition 4.1. If there exists an isomorphism φ of M_1 onto M_2 and an order-preserving injection $g: \overline{T}_1(U_1) \longrightarrow I$ such that

(1) $\varphi(M_1) = M_2$

(2) $g(u_1(x)) = u_2(\varphi(x)) \quad \forall\, x \in L(u_1)$

(3) $g(\overline{T}_1(a,b)) = \overline{T}_2(g(a),g(b)) \quad \forall\, a,b \in U_1$.

Then the anti T_1-fuzzy subgroup u_1 of G_1 and the anti T_2-fuzzy subgroup u_2 of G_2 are said to be isomorphic, and the pair (φ, g) is called an isomorphism of u_1 onto u_2.

Based on the properties of anti T-fuzzy subgroups, we have the following two basic isomorphism theorems for anti T-fuzzy subgroups:

Theorem 4.1. Let u_i be anti T_i-fuzzy subgroup of G_i $(i=1,2)$, φ an isomorphism of M_1 onto M_2. If T_1 is generated by T_2 via f, then $(\varphi, \overline{f}|_{\overline{T}_1(U_1)})$ is an isomorphism of u_1 onto u_2 iff $\overline{f}(u_1(x)) = u_2(\varphi(x)) \quad \forall\, x \in M_1$. Where $\overline{f}(t) = 1 - f(1-t) \quad \forall\, t \in I$, $\overline{f}|_{\overline{T}_1(U_1)}$ is the restriction of f to $\overline{T}_1(U_1) \subseteq I$.

Theorem 4.2. Let t-norm T_1 be generated by a t-norm T_2 via f, u_1 an anti T_1-fuzzy subgroup of G_1. If M_1 is isomorphic to a certain subgroup S_2 of G_2, then u_1 is isomorphic to an anti T_2-fuzzy subgroup of G_2.

We omit the proofs of theorem 4.1 and theorem 4.2. In theorem 4.2, if we take $G_1 = G_2 = G$, $S_2 = M_1$ and φ the identical automorphism of M_1, we get the following corollary:

Corollary 4.1. If t-norm T_1 is generated by a t-norm T_2, then each anti T_1-fuzzy subgroup of G is isomorphic to an anti T_2-fuzzy subgroup of G.

Presenting the examples of isomorphisms of anti T-fuzzy subgroups and the applications of corollary 4.1, we may prove that for any given t-norm $T \in (T_{SS} \cup T_H \cup T_F \cup T_Y \cup T_S) \setminus \{Min\}$, every anti T-fuzzy subgroup of a group G is isomorphic to a T_P-fuzzy subgroup of G or a T_M-fuzzy subgroup of G, wher T_{SS}, T_H, T_F, T_Y and T_S denote the Schweizer-Sklar's, Hamacher's, Frank's, Yager's and Sugeno's families of t-norms, respectively; and $T_P(a,b) = a.b$, $T_M(a,b) = \max(a+b-1,0)$, $\forall\ a,b \in I$.

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