Rule Interpolation by α-level Sets in Fuzzy Approximate Reasoning*

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1. Introduction

It is well known that even modern control theory has failed to cope with many control problems in industrial processes, vehicles, household equipment, etc. etc. The real problem is that classical control theory describes adequately only a limited class of not very complex systems. On the other hand, heuristic and intuitive control by a human operator often solves the problem of controlling very complex systems in a satisfactory degree. So e.g. driving a car can be solved by most grown-up people (at least after a period of appropriate training) but nobody could as far solve the fully automatic control of driving a car in a real traffic environment. This system seems to be too complicated to be modelled satisfactorily by any known mathematical method.

In recent years a good many of succesful control applications have invaded the market which have the common feature of using the idea of linguistic/approximate reasoning formalized by fuzzy rules and inference.

The idea of rule based fuzzy inference was proposed originally by Zadeh [1]. First applicational results were produced in laboratory environment by Mamdani and colleagues [e.g. 2]. In the last years center of gravity of applications has been shifted unambiguously to Japan where hundreds of real industrial apllications based on the reserach work done by Sugeno [3], Hirota [4] and others appeared in the middle of the 80's.

When using fuzzy inference algorithms to industrial control, one of the crucial problems is the computational speed of the applied method. Computational speed is mathematically described by algebraic complexity. Control methods having good sensitivity have also high complexity. The compact rule method reproposed by the Authors [5] combined with some boundedness type restrictions leads to an acceptable complexity [6,7,8] if the number of rules is not too high and especially if the support sizes in the rules and observations are small enough.

The above restrictions however may lead to a 'low density' of

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rules in the observation and conclusion space. This raises a new problem in obtaining well applicable control algorithms.

2. The problem of rule interpolation

Let us consider the problem with the generalized modus ponens illustrated with tomato colours and degree of ripeness by Zimmermann and Mizumoto [9,10]. We can compare three types of reasoning:

1. Simple modus ponens.

   'If a tomato is red then the tomato is ripe'
   'This tomato is red'

   'This tomato is ripe.'

2. Generalized modus ponens.

   'If a tomato is red then the tomato is ripe'
   'This tomato is very red'

   'This tomato is very ripe.'

3. Open problem - no conclusion.

   'If a tomato is red then the tomato is ripe'
   'If a tomato is green then the tomato is unripe'
   'This tomato is yellow'

   'This tomato is ???'

Let us illustrate reasoning type 3 by a simple figure using triangular membership functions (Fig. 1). Observation space X contains colours from a deep green to a deep red and linguistic fuzzy terms of colours can be introduced over this space: very green, green, greenish yellow, yellow, reddish yellow, red, very red. Conclusion space Y contains degrees of ripeness over which such terms can be introduced: unripe, almost unripe, little ripe, halfripe, quite ripe, almost ripe, ripe. Rules R_1 and R_2 are represented by membership function pairs in X and Y, resp., observation O is a membership function in X. There is no overlapping between 'yellow' and 'green', neither between 'yellow' and 'red'. So using the practical reasoning algorithms applied in the industry, the conclusion is a membership function identically 0, i.e. no conclusion whatever can be calculated. On the other hand we feel intuitively that a conclusion *'This tomato is halfripe'* would be reasonable.

A solution of this contradiction is the introduction of rule interpolation. Our observation is in some sense between the two 'if-parts' of rules R_1 and R_2 so we expect the conclusion also between the 'then-parts'. An exact formulation of this statement can be done by using rule interpolation.

3. Linear interpolation of two rules

Let us denote the 'if-parts' of the rules by I_1 and I_2, the 'then-parts' by T_1 and T_2, respectively. Linear interpolation of the two rules can be intuitively defined as follows:

$$\text{distance}(O, I_1) : \text{distance}(O, I_2) = \text{distance}(C, T_1) : \text{distance}(C, T_2)$$

where C is the conclusion. This is the philosophy of linear rule interpolation.

It is not very obvious what is the distance of two fuzzy terms. as e.g. what is distance(yellow,green). It is possible to introduce some measure in X and Y. Let us introduce X and Y as finite intervals $[x_1, x_{12}]$ and $[y_1, y_{12}]$ where for simplicity we use:

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\} \text{ and}$$

$$Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\},$$

however adding elements like $x_{4.5}$ etc. if necessary. Then the distances $d(y_i, y_j)$ and $d(x_i, x_j)$ can be defined as $|i-j|$.

These definitions however do not bring us really near to the idea of the distance of fuzzy terms. If the terms have uniform shape membership functions the distance can be understood as e.g. the distance of the maximums. In most of the application cases however uniformity (including shape and size) cannot be guaranteed.

We shall introduce another extension of the idea of distance. Membership function of a fuzzy set can be defined as

$\mu(A) = \bigcup_{\alpha \in [0,1]} \alpha.\chi(A_\alpha)$ where A_α is the α-level set of A. $\bigcup$ is understood as sup and A_0 stands for supp(A). Our proposal for the interpolation is that it is calculated on a finite set of α-levels and distance is calculated for every α-level independently.

Definition 1

Let A(x) and B(x) be two convex fuzzy sets in X, where $\#X < \infty$, $d(x_i, x_j) = |i-j|$ and $x_i < x_j$ iff $i < j$. Then the lower distance of A_α and B_α is

$$d_L(A_\alpha, B_\alpha) = d(\inf\{A\}, \inf\{B\})$$

and the upper distance of A_α and B_α is

$$d_U(A_\alpha, B_\alpha) = d(\sup\{A\}, \sup\{B\}).$$

In the practical applications instead of inf and sup, min and max can be used.

Let us define 'green' as

$\{0/x_1, 0.67/x_2, 0.67/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0/x_8, 0/x_9, 0/x_{10}, 0/x_{11}, 0/x_{12}\}$,

and 'yellow' as

$\{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.67/x_6, 0.67/x_7, 0/x_8, 0/x_9, 0/x_{10}, 0/x_{11}, 0/x_{12}\}$,

then $\text{green}_{0.67} = \{x_2, x_3, x_4\}$ and $\text{yellow}_{0.67} = \{x_6, x_7, x_8\}$. The lower and upper distances are

$$d_L(\text{green}_{0.67},\text{yellow}_{0.67}) = d(\min\{x_2,x_3,x_4\},\min\{x_6,x_7,x_8\}) = 4$$

$$d_U(\text{green}_{0.67},\text{yellow}_{0.67}) = d(\max\{x_2,x_3,x_4\},\max\{x_6,x_7,x_8\}) = 4$$

Let us return now to the intuitive formula for conclusion C. on level α we have

$$d_L(O_\alpha,I_{1\alpha}) : d_L(O_\alpha,I_{2\alpha}) = d_L(C_\alpha,T_{1\alpha}) : d_L(C_\alpha,T_{2\alpha}) \text{ and}$$

$$d_U(O_\alpha,I_{1\alpha}) : d_U(O_\alpha,I_{2\alpha}) = d_U(C_\alpha,T_{1\alpha}) : d_U(C_\alpha,T_{2\alpha}).$$

Solving the above equations for $\min\{C_\alpha\}$ and $\max\{C_\alpha\}$ we obtain

$$C_\alpha = \left[\frac{w^\alpha_{1L}.\min\{T_{1\alpha}\} + w^\alpha_{2L}.\min\{T_{2\alpha}\}}{w^\alpha_1 + w^\alpha_2}, \frac{w^\alpha_{1U}.\max\{T_{1\alpha}\} + w^\alpha_{2U}.\max\{T_{2\alpha}\}}{w^\alpha_1 + w^\alpha_2}\right],$$

where $w^\alpha_{1L} = d(\min\{O_\alpha\},\min\{I_{2\alpha}\})$ and $w^\alpha_{2L} = d(\min\{O_\alpha\},\min\{I_{1\alpha}\})$ and

$w^\alpha_{1U} = d(\max\{O_\alpha\},\max\{I_{2\alpha}\})$ and $w^\alpha_{2U} = d(\max\{O_\alpha\},\max\{I_{1\alpha}\})$ or

$w'^\alpha_{1L} = 1/w^\alpha_{2L}$ and $w'^\alpha_{2L} = 1/w^\alpha_{1L}$, similarly w'^α_{1U} and w'^α_{2U}, which lead to an equivalent result.

Let us calculate now the interpolation for the conclusion of reasoning 3. For α we choose the set {0, 0.67}. Then

$$C_0 = [(1/4.y_1+1/4.y_9).2,(1/4.y_4+1/4.y_{12}).2] = [y_5,y_8]$$

$$C_{0.67} = [(1/4.y_2+1/4.y_{10}).2,(1/4.y_3+1/4.y_{11}).2] = [y_6,y_7]$$

(If we wanted to calculate C_1 as well, it would require a denser scale in X and Y otherwise we would find that C_1 is the empty set.)

The reconstruction of C shows the membership function in Fig. 2, which is identical with the obvious definition of 'halfripe'. so we can reach a conclusion also in case 3:

'This tomato is halfripe'

Another example can be seen on Fig. 3, where although only triangular membership functions are applied but they have varying widths. (For simplicity only the indices are marked.) It is enough to calculate the support and the maximum as linear interpolation of triangular membership functions always leads to triangular results. This can be stated also in general:

Statement 1

With triangular membership functions in the rules and the observation the interpolated conclusion is also triangular and it is enough to calculate only two different α-s in order to reconstruct the conclusion.

In the example there are two rules which can be described briefly as

R_1: 1Δ3 ⇒ 8Δ12

R_2: 7Δ11 ⇒ 2Δ4 (iΔj stands for a symmetrical triangular membership function with support [i,j]). The observation is 4Δ6. Calculating the weights according to the primed versions we have:

$w^0_{1L} = 1/3$, $w^0_{2L} = 1/3$, $w^0_{1H} = 1/3$, $w^0_{2H} = 1/5$; $w^1_1 = 1/3$, $w^1_2 = 1/4$, so the conclusion is 5Δ9. (Validity of the Statement can be checked by calculations with support and maximum.)

In all the above examples one dimensional X and one dimensional Y were treated. In the practical application however usually the rules contain more then one fuzzy variables both in X and Y which means that observation and conclusion space are multidimensional. Rules have the form

'If x_1 is A_{1i} and x_2 is A_{2i} and ... and x_m is A_{mi}

then y_1 is B_{1i} and y_2 is B_{2i} and ... and y_n is B_{ni}'

In order to extend the above method for multidimensional rules (and observations) it is necessary to go back to the idea of distance between two fuzzy terms. Restricting the examination to an arbitrary α-level set the distances in all dimensions of X can be calculated separately as it was done in one dimension. So we obtain m distance pairs $d^\alpha_{11}, d^\alpha_{12}, \ldots, d^\alpha_{1m}$ and $d^\alpha_{21}, d^\alpha_{22}, \ldots, d^\alpha_{2m}$ (cf. Fig. 4).

For the interpolation a pair of single weighting factors is necessary which must somehow accumulate the information in all the distance pairs. we propose the reciproc value of the length of the 'vector' calculated by taking the distances as components. It is supposed however that a uniform measure is introduced over all the componenets of X. By varying the measure scale over the various dimensions the relative weight of them can be changed. Our proposal for the weights is:

$$w^\alpha_{1L} = \left(\sqrt{(d^\alpha_{11})^2+(d^\alpha_{12})^2+\ldots+(d^\alpha_{1m})^2}\ \right)^{-1}, \text{ etc.}$$

In all dimensions of Y this weights are used in order to calculate the minimum and maximum of the α-level sets.

Finally, it is necessary to deal with some extremal cases. If the distance is 0, the corresponding weight will become '∞'. This means that the other rule will play no role in the calculation of conclusion. This can be formulated in a Statement.

Statement 2

If the observation is identical with the 'if-part' of rule R_i then the conclusion will be identical with the 'then-part' of the same rule (T_i).

Another extremal case is if the membership function has a positive value at the 'end' of X. Then, an 'extrapolation' of the function is advisable, e.g. resulting into 'negative indices' of x, an S-shaped membership function can be represented e.g. by an imaginary support limit in $-\infty$ or $+\infty$. The use of such corrections of the method requires some more theoretical and practical investigations.

4. General interpolation

In Section 3 only the case of interpolation on the basis of two rules was treated. However, the idea used there can be extended for an arbitrary number of rules, supposing that the observation is flanked from both sides by at least one rule. If distance-reciproc weights are used the proportional role of each rule is guaranteed.

It is interesting to check what is happening in the case of several (more than 2) rules in the context of Statement 2. As Rules are different from each other, it can be also supposed that their 'if-parts' are different. So there is maximally one rule for which the 'if-part' is identical with a given observation. In that case however the weight attached to this rule will be overwhelming in comparison with all the other rules and so the conclusion must be identical with the 'then-part' of the mentioned rule.

Also linearity is not necessarily kept as a rule. Rules located far away often are not very important for a given observation so e.g. the square of the distance can be used as reciproc weight. Every kind of interpolation where the weights obtained by some mapping of the 'distance of two membership functions' (in the sense as it was introduced in this paper) and the following properties hold:

$w(d)$ is a monotonously decreasing function of d in $[0,+\infty)$

$w(0) \to \infty$

$\lim_{d \to \infty} w(d) = 0$

Statement 2 will be valid.

A problematic question is the inhomogeneity of measure over various dimensions in the observation space. A way out is presented by general normation. Fuzzy terms over a given fuzzy variable are usually expressing different fuzzy degrees of a given fact. In our example with tomatoes it is known that the colour of a tomato varies from deep green to deep red during the phases of its ripening. This offers the possibility of projecting the given finite scale of colours into e.g. the interval [0,1], so that 'very green' starts with 0 and 'very red' ends with 1. Then the term 'yellow' will be described e.g. by $0.4\Delta 0.6$. Similarly the size of a tomato will vary from 'very small' to 'very large' (e.g. 2cm to 10cm diameter) and this size is also normed i.e. projected to [0,1]. Then the 'Euclidean distance' of the minimums of two α-level sets (calculated by the square root of the sum of component distance squares) has some meaning easier to understand in a given application context.

It is not without interest to examine the problem of complexity in connection with interpolation. It is easy to prove that in case of bounded rules the interpolated conclusion will be also bounded by the same upper bound. This fact enables the use of bounded compact rule reasoning algorithm over a bounded subspace in XxY with no dependence on the number of rules, when calculating the center of gravity (cf. [8]).

The use of rule interpolation opens some new possibilities to industrial application of fuzzy control.

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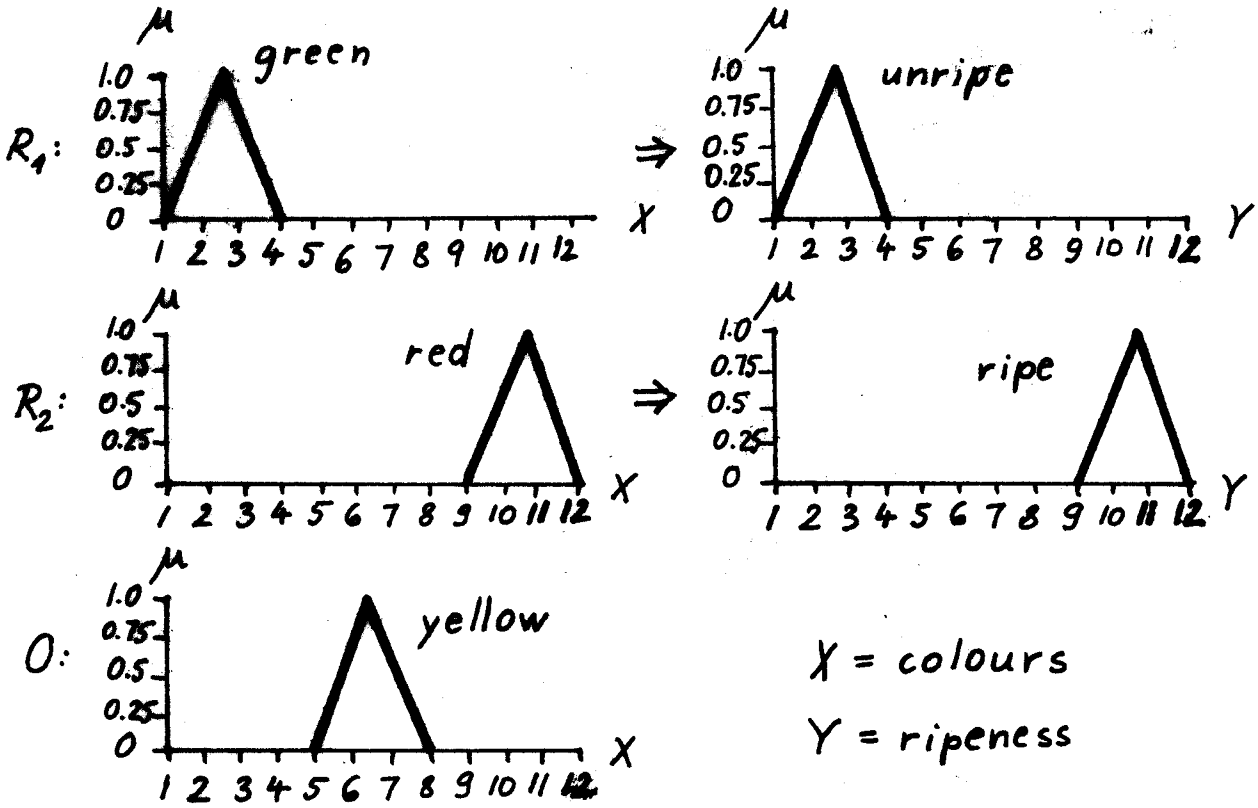

Figure 1.

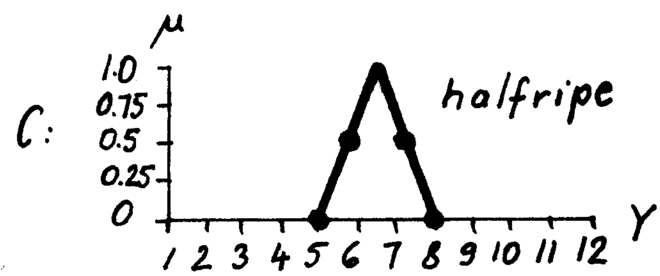

Figure 2.

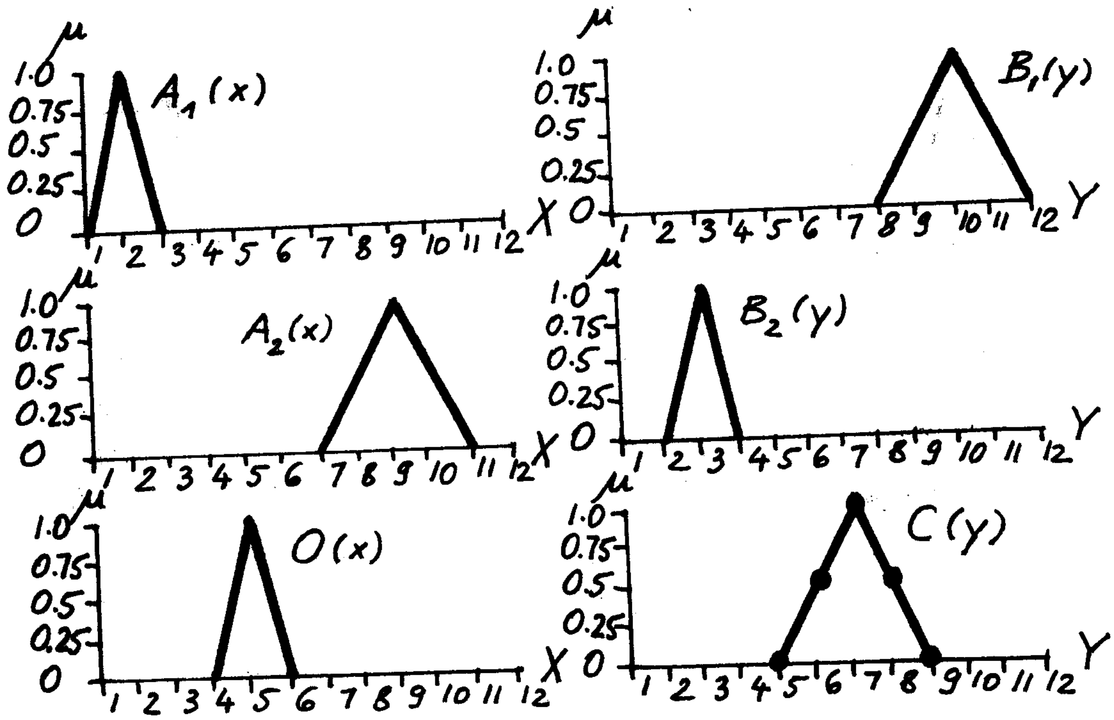

Figure 3.

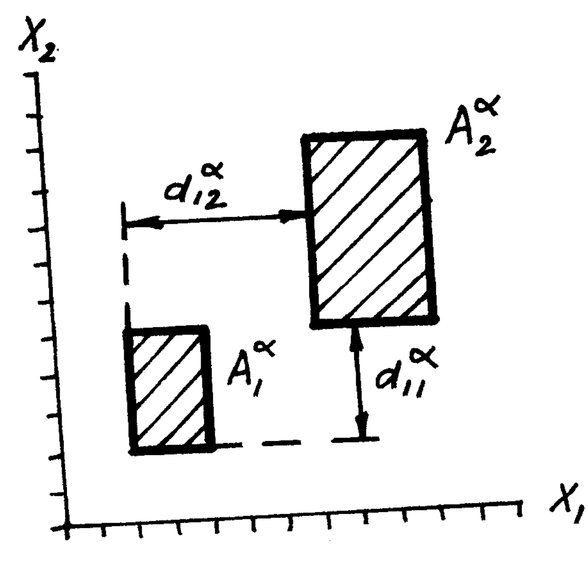

Figure 4