# FUZZY SUBALGEBRA OF A UNIVERSAL ALGEBRA

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Abstract: The object of this paper is to study fuzzy subsystem of a universal algebra. In section 3, a fuzzy subalgebra(fsa) of a universal algebra(u.a) under tringular norm is defined and some properties are derived.Level subalgebras of a fsa's are dealt with in section 4. Some results on union of two fsa's are established in section 5. In particular,a sufficient condition for the expressibility of a u.a. as the union of two proper fsa's is obtained. Subalgebra generated and function generated fsa are introduced in section 6 and results analogous to these in the case of groups are obtained. Some results on fuzzy relations and fuzzy subalgebras are given in section 7.

Some important definitions and results (without proof) are given below.

Unless otherwise mentioned in what follows $(G,\Omega)$ will denote a u.a., A,a fuzzy subset of G and T,an arbitrary t-norm. Also the universal algebras,considered, are assumed to be of the same type.

Definition 1: A is said to be a T-fuzzy subalgebra (T-fsa) of G if for all n-ary operation $\omega \in \Omega$ and $a_1, a_2, \ldots, a_n \in G$

$$A(a_1 a_2 \ldots a_n \omega) \geqslant T(A(a_1), \ldots, A(a_n)).$$

If T = M, then A is said to be a fuzzy subalgebra (fsa) of G.

Remark 2: Every fsa of G is a T-fsa of G but example is given to show that the converse is not true.

Theorem 3: The characteristic function $\chi_H$ of a subset H of $(G, \Omega)$ is a T-fsa iff H is a subalgebra of G.

Theorem 4: The arbitrary intersection of T-fsas is a T-fsa.

Theorem 5: The homomorphic image of a T-fsa, when T is continuous is a T-fsa. The homomorphic pre-image of a T-fsa is also a T-fsa.

Theorem 6: Let $B_i$ be T-fsa of $(G_i, \Omega)$ for $i = 1, 2, \ldots, s$. $B = B_1 \otimes_{T'} \ldots \otimes_{T'} B_s : G_1 \times G_2 \times \ldots \times G_s$ ----->I defined by

$$B(x_1, \ldots, x_s) = T'(B_1(x_1), \ldots, B_s(x_s)) \quad \forall\, x_i \in G_i$$

is a T-fsa of $G_1 \times \ldots \times G_s$ if $T'$ dominates T.

Remark 7: If $\exists\, e_i \in G$ s.t. $e_i \ldots e_i \omega = e_i$ and $B_i(e_i) = 1$ for $i = 1, 2, \ldots, s$, then the converse is also true.

Remark 8: Let $\Omega$ consist of only one n-ary operation $\omega$. Let $H_1, \ldots, H_n$ be subalgebras of G s.t. every $x \in G$ can be uniquely expressed as $x = x_1 \ldots x_n \omega$ where $x_i \in H_i$. Let B, $B_i$ be T-fsas of G, $H_i$ respectively. Let $\hat{B}_i, \check{B}_i$ :G----->I be defined by

$$\hat{B}_i(x) = B(x_i) \text{ and } \check{B}_i(x) = B_i(x_i) \text{ where } x = x_1 \ldots x_n \omega$$

If $(G, \Omega)$ be a group, then $\hat{B}_i$, $\check{B}_i$ are T-fuzzy subgroups of G. However this is not true if G be an arbitrary u.a. But if $\omega$ satisfies the generalised medial law

$$(a_{11} a_{12} \ldots a_{1n}\omega)(a_{21} a_{22} \ldots a_{2n}\omega) \ldots (a_{n1} a_{n2} \ldots a_{nn}\omega)\omega$$

$= (a_{11}\ a_{21} \ldots a_{n1}\omega) \ldots (a_{1n}\ a_{2n} \ldots a_{nn}\omega)\ \omega$

then $\hat{B}_i$ and $\check{B}_i$ are T-fsa^ of G.

Theorem 9: If B be a fsa of $(G,\Omega)$ then $B_t = \{x \in G, B(x) \geq t\}$ is empty or a subalgebra $\forall t \in I$. $B_t$ is called level subalgebra of B.

Remark 10: (i) However the above is not true if B is a T-fsa of G where $T \neq M$.

(ii) Two different fsa^ may have the same family of level subalgebras.

Theorem 11: Given any chain of subalgebras $G_0 \subset G_1 \subset \ldots \subset G_r = G$ of $(G,\Omega)$, $\exists$ a fsa of G whose level subalgebras are precisely the members of the chain.

Remark 12: The union of two T-fsa^ may not be a T-fsa.

Remark 13: A u.a $(G,\Omega)$ can be expressed as $A \cup B$ where A, B are two fsa^ s.t $A \nleq B$ or $B \nleq A$.

Remark 14: Rosenfeld [3] proved a group can not be expressed as the union of two proper fuzzy subgroups. In the case of universal algebra, we have the following result

Theorem 15: Let $\exists$ an n-ary operation $\omega \in \Omega$ s.t.

(i) $\omega$ is associative,

(ii) $\exists$ an element $e_\omega \in G$ s.t. $\forall a \in G$,

$\omega(e_\omega, \ldots, \overset{i}{a}, \ldots e_\omega) = a\ \forall i = 1,2,\ldots,n$

(iii) $\forall b \in G\ \exists\ b_1 \in G$ s.t. $\omega(e_\omega, \ldots, e_\omega, \overset{i}{b}, e_\omega, \ldots e_\omega, \overset{j}{b_1}, e_\omega, \ldots e_\omega) = e_\omega$

for $i,j = 1,2,\ldots,n$ amd $i \neq j$.

Let B and A be two proper fsa^ of $(G,\Omega)$ s.t. $B(e_\omega) = A(e_\omega) = 1$ and $B(b) = B(b_1)$, $A(b) = A(b_1)$ $\forall b \in G$. Then G can not be expressed as the union of B and A.

Following the concept of 'subgroup generated' and 'function generated' fuzzy subgroups introduced by Anthony and Sherwood [2], the concept of subalgebra generated and function generated fuzzy subalgebras have been defined and results analogous to those obtained by Anthony and Sherwood have been obtained .

Remark 16: Let B be a fsa of GxG. The weakest fuzzy subset of G on which B is a fuzzy relation is denoted by $A_B$ and is defined by

$$A_B(x) = \sup \{\max\{B(x,y), B(y,x);\ y \in G\}\}.$$

Now the problem whether $A_B$ is a fsa is remains open . However, we have,

Theorem 17: Let $(G, \Omega)$ be a u.a and B be a fsa of GxG s.t. $\forall\ x \in G$ $\sup\{B(x,y);\ y \in G\} = \sup\{B(y,x);\ y \in G\}$ ... ...(1)

Then $A_B$ is also a fsa of G.

Remark 18: Example has been given to show that the condition (1) is not necessary.

Theorem 19: Let B be a fsa of $(G\times G, \Omega)$. If for each $x \in G$ and n-ary operation $\omega \in \Omega$ there exist $x_1, x_2, \ldots, x_n \in G$ s.t.

$$x_1 x_2 \ldots x_n \omega = x \quad \ldots \quad \ldots \quad \ldots \quad \ldots(2)$$

$$\text{and } B(x_i, t) \geqslant B(x,t) \ \forall\ t \in G \quad \ldots \quad \ldots \quad (3).$$

Then $A^B_x$ is a fsa of G where $A^B_x(y) = B(x,y)$

Remark 20: The conditions (2) & (3) are satisfied if G be a group.

## REFERENCES


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