L-fuzzy Primary and Semiprime Ideals

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In this note we introduce a definition for L-fuzzy primary ideal, in term of L-fuzzy points, and prove that this definition is equivalent to the definition 5.1 of [3], where L=[0,1]. Moreover L-fuzzy semiprime ideals are introduced and some equivalent conditions of a L-fuzzy semiprime ideal are proved.

Keywords: L-fuzzy prime, primary, maximal, radical, and semiprime ideal.

1. Introduction

In [6] a definition of L-fuzzy prime ideal is given,which is not suitable one, by Theorem 2.1 of [5]. Similary in Definition 3.1 of [6], L-fuzzy primary ideal is defined which is not suitable by Theorem 5.9 of [3], since in this definition the cardinality of the image of a L-fuzzy primary ideal could be more than 2. Hence Theorem 3.1 of [6] and Definition of P-Primary ideal are not correct. Fortunately in [3] for the case L equal to the interval [0,1], an elegant definition of fuzzy primary ideal has been given.

In this note the L-fuzzy primary ideals is redefined, in term of L-fuzzy points, and it is proved that this definition is equivalent to that of [3, Definition 5.1] for the case L equal to [0,1]. Also we see that there is no need for the restrictive condition $P_* \supset Q_*$ imposed on definition of fuzzy radical ideal in [3, Definition 4.3], which is removed in this note, and some results on L-fuzzy primary ideal are given.

Moreover L-fuzzy semiprime ideals are introduced and some equivalent conditions of a L-fuzzy semiprime ideal are proved. It is noted that the L-fuzzy semiprime ideals can not be characterized in the form of L-fuzzy prime, maximal and primary ideals as in Theorems 2.1, 3.1 of [5], Theoremes 5.9, 5.10 of [3] and Theorem 3.3 of this note, respectively. But a characterization of semiprime ideals when R is commutative is given.

2. Preliminaries

We fix $L=(L,\leq,\vee,\wedge)$ as a completely distributive lattice, with the least element 0 and greatest element 1. We write "sup" and "inf" for "$\vee$" and "$\wedge$" respectively. For a nonempty set X, let $F(X)=\{A \mid A$ is a L-fuzzy subset of $X\}$, then for $A,B\in F(X)$, we say that $A\subseteq B$ iff $A(x)\leq B(x)$; for all $x\in X$, and $A\subset B$ iff $A\subseteq B$ and $A\neq B$. By a L-fuzzy point x_r of X; $x\in X$, $r\in L$, we mean $x_r\in F(X)$ defined by $x_r(y)=\begin{cases} r & \text{if } y=x \\ 0 & \text{if } y\neq x \end{cases}$ and we write $x_r\in X$. If x_r is a L-fuzzy point of X and $x_r\subseteq A\in F(X)$, then we write $x_r\in A$. If $A\subseteq X$ we mean the characteristic function $\chi_A\in F(X)$, defined by $\chi_A(x)=\begin{cases} 1 & \text{if } x\in A \\ 0 & \text{if } x\notin A \end{cases}$.

From now on R is a ring.

Definition 2.1. Let $A\in F(R)$, then A is called a L-fuzzy left(right) ideal of R if and only if for all $x,y\in R$;

(i) $A(x-y) \geq \inf(A(x),A(y))$

(ii) $A(xy) \geq A(y)$ $\quad (A(xy) \geq A(x))$

A is called a L-fuzzy ideal of R iff it is both L-fuzzy left and L-fuzzy right ideal of R. We let $I_l(R)$, $I_r(R)$, $I(R)$ be the set of all L-fuzzy left, L-fuzzy right, and L-fuzzy ideals of R, respectively.

Definition 2.2. Let $A,B\in F(R)$ we define the L-fuzzy subsets AoB, AB as [5].

Definition 2.3. Let $A\in F(R)$, then $\langle A\rangle\in I(R)$, is defined by

$$\langle A\rangle(W)=\inf_{A\subseteq B\in I(R)} B(W)$$

Clearly $\langle A\rangle\in I(R)$, and it is called the L-fuzzy ideal generated by A.

Lemma 2.4 [7, Theorem 4.4]. Let R be a commutative ring with identity then

$$\langle x_r\rangle(W)=\begin{cases} r & \text{if } W=xy \text{ for some } y\in R \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.5. Let $\mu\in F(R)$, and $t\in L$, then the set $\mu_t=\{x\in R|\mu(x)\geq t\}$ is called a level subset of R with respect to μ.

Notation: Let $A\in I(R)$, we mean A_* is the level subset $A_{A(0)}=\{x\in R|A(x)\geq A(0)\}$.

Definition 2.6 [5, Definition 2.1]. A nonconstant L-fuzzy ideal P of R is called prime iff for any L-fuzzy ideals $A,B,AB\subseteq P$, implies either $A\subseteq P$ or $B\subseteq P$.

Definition 2.7 [5, page 101]. By a L-fuzzy maximal ideal of R, we mean, a maximal element in the set of all nonconstant fuzzy ideals of R, under the pointwise partial ordering.

Definition 2.8. An ideal $p\neq R$ is called semiprime, iff $a^2\subseteq p$ implies $a\subseteq p$, for every ideal a of R.

Notation: We mean Rad a is the radical of a; for any ideal a of R.

3. L-fuzzy primary ideal

In this section R is a commutative ring with identity.

Definition 3.1. Let $Q\in I(R)$ be nonconstant, Q is called

L-fuzzy primary ideal iff for every two L-fuzzy points $x_r, y_s \in R$, $x_r y_s \in Q$ implies that either $x_r \in Q$ or $y_s^n \in Q$, for some $n \in \mathbb{N}$.

Corollary 3.2. If P is a L-fuzzy prime ideal, then P is a L-fuzzy primary ideal.

Theorem 3.3. (a) Let q be a primary ideal and α a prime element of L then the L-fuzzy subset Q defined by

$$Q(x)=\begin{cases} 1 & \text{if } x\in q \\ \alpha & \text{otherwise} \end{cases} \qquad (1)$$

is a L-fuzzy primary ideal

(b) Conversely any L-fuzzy primary ideal Q, can be obtained as (1).

Remark 3.4. Let Q be a L-fuzzy primary ideal and $q=Q_*$. Let p=Rad q, then p is a prime ideal. If we consider the L-fuzzy ideal

$$P(x)=\begin{cases} 1 & \text{if } x\in p \\ \alpha & \text{otherwise} \end{cases}$$

where $\alpha\in L$ obtained by (1), then by Theorem 2.1 of [5] P is a L-fuzzy prime ideal. Moreover $P \geq Q$ and $P_*=p=$ Rad $q \geq Q_*$'.

Definition 3.5. Let $I\in I(R)$, we define Rad $I\in I(R)$ as follows:

$$\text{Rad } I=\begin{cases} \bigcap\limits_{P\geq I} P & \text{if There exists some L-fuzzy prime ideal } P\geq I \\ \chi_R & \text{otherwise} \end{cases}$$

Rad I is called the L-fuzzy radical of I.

Note: This definition is similar to radical of an ideal in ordinary ring theory, and it has not the restricted condition $P_* \geq I_*$ as it used in Definition 4.3 of [3], so Rad $I\subseteq \sqrt{I}$, where $\sqrt{I}$ is defined in Definition 4.3 of [3].

Theorem 3.6. Let Q be a L-fuzzy primary ideal and P be the L-fuzzy prime ideal which obtained in Remark 3.4, then Rad Q=P.

Definition 3.7. Let Q and P be as in above theorem, then we say that Q is P-primary.

Theorem 3.8. Let L=[0,1], then the definition of L-fuzzy primary ideal of this note is equivalent to the definition 5.1 of [3], when Q is nonconstant.

Proposition 3.9. Let M be a L-fuzzy maximal ideal of R, then for all $n \in \mathbb{N}$, M^n is M-primary.

Proposition 3.10. Let Q is P-primary, and x_r, y_s be two L-fuzzy points such that $x_r y_s \in Q$ and $x_r \notin P$, then $y_s \in Q$.

4. L-fuzzy semiprime ideal

Definition 4.1. A nonconstant L-fuzzy ideal P of a ring R is called semiprime ideal iff, for any fuzzy ideal A of R, $AA=A^2 \subseteq P$, implies $A \subseteq P$.

Note. (i) $A^2 \subseteq P$ iff $AoA \subseteq P$, so P is a L-fuzzy semiprime ideal iff $AoA \subseteq P$, implies $A \subseteq P$.

(ii) if P is a L-fuzzy semiprime ideal, then $A^n \subseteq P$, implies $A \subseteq P$, for all $n \in \mathbb{N}$.

(iii) Every L-fuzzy prime ideal is semiprime.

Theorem 4.2. If $P \in I(R)$, all of the following conditions are equivalent:

(i) P is L-fuzzy semiprime ideal

(ii) If $x_r \in R$ is such that $x_r \chi_R x_r \leq P$, then $x_r \in P$

(iii) If for $x_r \in R$, $\langle x_r \rangle^2 \leq P$, then $x_r \in P$

(iv) If $U \in I_r(R)$ such that $U^2 \leq P$, then $U \leq P$

(v) If $V \in I_l(R)$ such that $V^2 \leq P$, then $V \leq P$

Proposition 4.3. Let p be a semiprime ideal and $\alpha \neq 1$ be an arbitrary element of L, then the L-fuzzy subset P is a semiprime, where P is defined as follows:

$$P(x)=\begin{cases} 1 & \text{if } x \in p \\ \alpha & \text{otherwise} \end{cases}$$

Remark 4.4. The following example shows that the converse of above proposition is not true. So it is not a characterization of L-fuzzy semiprime ideals, where as for L-fuzzy prime, maximal, primary ideals there are characterization as in Theorem 2.1, 3.1 of [5] and Theorem 3.3 of this note.

Example 4.5. Let $R=\mathbf{Z}$ and $L=[0,1]$, Define P as follows:

$$P(x)=\begin{cases} 2/3 & \text{if } x=0 \\ 1/2 & \text{if } x \in 2\mathbf{Z}-\{0\} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $P \in I(R)$. Now if $A^2 \leq P$ for some $A \in I(\mathbf{Z})$, thensince $P(n^2)=P(n)$, for all $n \in \mathbf{Z}$, we get

$A(n)=\inf(A(n),A(n)) \leq AoA(n^2) \leq P(n^2)=P(n)$ for all $n \in \mathbf{Z}$. So $A \leq P$.

Note: If R is a commutative ring then there is a characterization for L-fuzzy semiprime ideal as in the following form.

Theorem 4.6. Let P be a nonconstant Lfuzzy ideal. Then if P is a L-fuzzy semiprime, then P_t is a semiprime ideal, for all

$t \in L$, which $P_t \neq \emptyset, R$. And conversely if R is commutative, and for all $t \in L$ $P_t \neq \emptyset, R$ is a semiprime ideal of R, then p is a L-fuzzy semiprime.

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