Some Correspondence Theorems on t-fuzzy subgroups

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Dedicated to Alireza Afzalipour the founder of Kerman University

Abstract

Recently some authors have considered the notion of t-fuzzy subgroup and proved some basic results (see references). Here we will prove some correspondence theorems for t-fuzzy (normal) subgroups.

Keywords: t-norm, t-fuzzy (normal) subgroup.

1. Preliminaries

From now on T is a continuous t-norm, unless otherwise stated, χ stands for the characteristic funtion, and G, G′ are groups with identity elements e, e′, respectively. If μ, η are fuzzy subsets of a non-empty set S, we write, $\mu \le \eta$ iff $\mu(x) \le \eta(x)$ for all $x \in S$.

Definition 1.1. Let T be an arbitray t-norm. The fuzzy subset μ of G is called t-fuzzy subgroup of G, iff (*i*) $\mu(xy) \geq T(\mu(x), \mu(y))$; $\forall$ $x, y \in G$.

(*ii*) $\mu(x^{-1}) = \mu(x)$; $\forall$ $x \in G$.

we denote it by $\mu <_T G$.

Definition 1.2. Let $\mu <_T G$. Then μ is called a t-fuzzy normal subgroup of G, iff $\mu(xy) = \mu(yx)$ for all $x, y \in G$. We denote it by $\mu \lhd_T G$.

A justification for Definition 1.2 can be obtained from the following theorem.

Theorem 1.3 [9, Theorem I.2.18]. Let $\mu <_T G$. Then the following conditions are equivalent:

(i) $\mu \lhd_T G$

(ii) $\mu(xyx^{-1}) = \mu(y)$; for all $x, y \in G$

(iii) $\mu(xyx^{-1}) \geq \mu(y)$; for all $x, y \in G$.

2. Some related results

In order to prove the correspondence theorems in section 3, we give some related results.

Theorem 2.1. Let $f: G \longrightarrow G'$ be a homomorphism, then

(i) if $\mu <_T G$, then $f(\mu) <_T G'$,

(ii) if $\mu \lhd_T G$ and f is surjective, then $f(\mu) \lhd_T G'$.

(iii) if $\chi_{Ker\ f} \subseteq \mu$, then $f(\mu)(e') = 1$.

Theorem 2.1(i) under sup property condition, has been given for t-fuzzy subfields in Proposition 3.5 of [3], and for L-fuzzy subgroups in Theorem 2 of [6].

Corollary 2.2. Let $N \lhd G$ and $\mu <_T G$ ($\mu \lhd_T G$). Then $\bar{\mu} <_T G/N$ ($\bar{\mu} \lhd_T G/N$), where

$$\bar{\mu}(\bar{g}) = \sup_{x \in \bar{g}} \mu(x); \ \forall \ \bar{g} \in G/N.$$

Moreover, if $\chi_N \subseteq \mu$, then $\bar{\mu}(N)=1$

The following corollary shows that Corollary 2.2 is a generalization for the normal part of Theorem 3.11 of [1], and Theorem 2.4 of [5].

Corollary 2.3. Let N, μ, and $\bar{\mu}$ be as in Theorem 3.11 of [1]. Then $\bar{\mu} \lhd_{Min} G$, and it is equal to that defined in Corollary 2.2.

In the following lemma by f-invariant we mean as in [7, p.515].

Lemma 2.4. Let $f: G \to G'$ be a homomophism, T be arbitrary, and $\mu' <_T G'$ ($\mu' \lhd_T G'$). Then $f^{-1}(\mu') <_T G$ ($f^{-1}(\mu') \lhd_T G$), and it is f-invariant. Moreover, if $\mu'(e')=1$, then $\chi_{\mathrm{Ker}\ f} \subseteq f^{-1}(\mu')$.

Also see Proposition 3.5 of [3].

Remark 2.5. Since in Theorems 3.9 and 3.12 of [1], one can write $\lambda = f^{-1}(\mu)$, where f is ϕ or θ, so Lemma 2.4, generalizes these theorems.

Corollary 2.6. Let $N \lhd G$, and $\mu' \underset{T}{<} G/N$ ($\mu' \underset{T}{\lhd} G/N$), and T be arbitrary. Then $\mu \underset{T}{<} G$ ($\mu \underset{T}{\lhd} G$), where

$$\mu(g)=\mu'(gN);\ \forall\ g \in G.$$

Moreover, if $\mu'(N)=1$, then $\chi_N \subseteq \mu$.

Lemma 2.7. Let $f:G \longrightarrow G'$ be an epimorphism, $\mu \underset{T}{<} G$, $\chi_{Ker\ f} \subseteq \mu$ and T be arbitrary. Then $f^{-1}(f(\mu))=\mu$.

3. Correspondence theorems

Theorem 3.1. Let $f:G \longrightarrow G'$ be an epimorphism. Then there is a bijedtion ψ between the set $\mathfrak{F}$ of all t-fuzzy (normal) subgroups of G containing χ_{Kerf} and the set $\mathfrak{F}'$ of all t-fuzzy (normal) subgroups of G' that are equal to 1 on e'.

Corollary 3.2. For a group G, there is a bijection between the the set of all t-fuzzy (normal) subgroups of F containing $\chi_{Ker\ f}$ and the set of all t-fuzzy (normal) subgroup of G that are equal to 1 on e, where $f:F \longrightarrow G$ is suitable epimorphism for a well chosen free group F.

Corollary 3.3. Let $N \lhd G$. Then there is a bijection ψ between the set $\mathfrak{F}$ of all t-fuzzy (normal) subgroup of G containing χ_N and the set $\mathfrak{F}'$ of all t-fuzzy(normal) subgroup of G/N that are equal to 1 on N.

Theorem 3.4. Let $f:G \to G'$ be an isomorphism, and T be arbitrary. Then there is a bijection ψ between the set $\mathfrak{F}$ of all t-fuzzy (normal) subgroups of G and the set $\mathfrak{F}'$ of all t-fuzzy (normal) subgroups of G'.

Lemma 3.5. Let $f:G \to G'$ be an epimorphism. Then $\mathfrak{F}' = \mathcal{D}'$, where

(*i*) $\mathfrak{F}' = \{\mu' \mid \mu' <_T G'\}$, $\mathcal{D}' = \{f(\mu) \mid \mu <_T G\}$,

(*ii*) $\mathfrak{F}' = \{\mu' \mid \mu' \lhd_T G'\}$, $\mathcal{D}' = \{f(\mu) \mid \mu \lhd_T G\}$.

Corollary 3.6. Let $N \lhd G$ and $f:G \to G/N$ be the canonical homomorphism. Then $\mathfrak{F}' = \mathcal{D}'$, where

(*i*) $\mathfrak{F}' = \{\mu' \mid \mu' <_T G/N\}$, $\mathcal{D}' = \{\bar{\mu} \mid \bar{\mu}$ as in Corollary 2.2, $\mu <_T G\}$,

(*ii*) $\mathfrak{F}' = \{\mu' \mid \mu' \lhd_T G/N\}$, $\mathcal{D}' = \{\bar{\mu} \mid \bar{\mu}$ as in Corollary 2.2, $\mu \lhd_T G\}$,

Corollary 3.7. Let $f:G \to G'$ be an epimorphism. Then there is a bijection between the set of all t-fuzzy (normal) subgoups of G' and the set of all t-fuzzy (normal) subgroups $\bar{\mu}$ of G/Ker f as defined in Corollary 2.2.

Note that the above corollary extends Theorem 3.9 of [8].

Acknowledgement: The authors would like to thak the Institute for studies in Theoretical physics and Mathematics (Iran) for their financial support.

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