Grey Limit -- Limit of Complex Fuzzy Function

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Problems concerning grey functions (or complex fuzzy functions) have been dealt with in our previous paper "Grey Function - Complex Fuzzy Function". The concept of limit in classical mathematics is based on functions. Now, on the basis of grey functions we would like to further explain the problems related to grey limit (or complex fuzzy limit).

The concept of limit is one of the most important concepts in calculus. Here we extend it to the classical rational grey numbers.

Concept of Grey Limit

Definition 1 Let D be the subset of the set of classical rational grey numbers $D \subseteq R$, and x_0 be an arbitrary classical grey number $x \in R$. If for any given real number $\varepsilon > 0$, there always exists $x' \in D$ and $x' \neq x_0$, such that $d(x', x_0) < \varepsilon$, then x_0 is called the point of accumulation of D.

Definition 2 For grey function $y = f(x)$, $x \in D$, let $x_0 \in R$ be an accumulation point of D, and A be a classical rational grey number. If for any given real number $\varepsilon > 0$, there always exists some real number $\delta > 0$ so that for $x \in D$, if $x \neq x_0$ and $d(x, x_0) < \delta$, then $d(f(x), A) < \varepsilon$ holds, then we can say when x approaches x_0 along D, the limit of f(x) is A, and is written as $\lim\limits_{x \xrightarrow{D} x_0} f(x) = A$ or $f(x) \to A(x \xrightarrow{D} x_0)$.

By definition, it can be seen that the limit of function f(x) is discussed when independent variable x approaches the accumulation point of D along the domain of definition D, and usually D can be neglected. Then, the limit is written as $\lim_{x \to x_0} f(x) = A$.

Example 1 Prove that for arbitrary $x_0 \in R$, grey function $y = \underset{\sim}{[a,b]}$ $x \in D = R$ has the following:

$$\lim_{x \to x_0} y = \underset{\sim}{[a,b]}$$

Proof:

If for any given real number $\varepsilon > 0$, we take $\delta = \varepsilon > 0$, then when $d(x, x_0) < \delta = \varepsilon$, $d(\underset{\sim}{[a,b]}, \underset{\sim}{[a,b]}) = 0 < \varepsilon$ will be true. By definition of limit, $\lim_{x \to x_0} \underset{\sim}{[a,b]} = \underset{\sim}{[a,b]}$

As a rule, for the constant grey function $y = A$, $D = R$, here A is a definite classical grey number, we have $\lim_{x \to x_0} y = A$.

Example 2 Prove that if the grey function $y = f(x) = x$, $x \in D = R$, then, there will be:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

Proof: If for any given real number $\varepsilon > 0$ we take $\delta = \varepsilon > 0$, then, when $d(x, x_0) < \delta = \varepsilon$, $d[f(x), x_0] = d(x, x_0) < \varepsilon$ will be true. Therefore, $\lim_{x \to x_0} f(x) = x_0$.

Definition 3 If $y = f(x)$, $x \in D$, and $\lim_{x \to x_0} f(x) = f(x_0)$ holds, then we say $f(x)$ is continuous at x_0 with respect to D.

Example 1 $y = A$ (a constant), $D = R$ is continuous for arbitrary $x_0 \in R$ with respect to D.

Example 2 $y = f(x) = x$, $D = R$ is also continuous for arbitrary $x_0 \in R$ with respect to D.

Definition 4 Let $u(x)$ be an arbitrary classical rational grey number with $P[u(x)]$ and $Q[u(x)]$ standing for its left and right end points respectively. Then for the given grey function $y = f(x)$, $x \in D$, there are four definite functions: $P[f(x)]$, $Q[f(x)]$, $\inf f(x)$ and $\sup f(x)$ with their domain of definitions being also D. They are called the left and right end points, the inferior and the superior limit functions of $y = f(x)$, respectively.

Here we would point out that sup $f(x) \equiv 1$, inf $f(x) = 1$ or 0, and both $P[f(x)]$ and $Q[f(x)]$ are the grey functions which take only real values. As a classical rational grey number $u(x)$ is determined solely by its $P[u(x)]$, $Q[u(x)]$, and inf $u(x)$, they can be regarded as three-variable real functions in classical calculus, that is, single-valued real functions with three independent variables, and one of the variables, inf $u(x)$, takes either 0 or 1 out of the two values.

Theorem 1 If $\lim_{x \to x_0} f(x) = A$,

then, $\lim_{x \to x_0} P[f(x)] = P[A]$

$\lim_{x \to x_0} Q[f(x)] = Q[A]$

$\lim_{x \to x_0} \inf f(x) = \inf A$

Proof: For any given $\varepsilon > 0$, since $\lim f(x) = A$, there exists $\delta > 0$, and when $d(x, x_0) < \delta$, we have

$d(f(x), A) < \varepsilon$,

that is, $\operatorname{Max}\{|P[f(x)] - P[A]|, |Q[f(x)] - Q[A]|, |\inf f(x) - \inf A|\} < \varepsilon$

so, $|P[f(x)] - P[A]| < \varepsilon$, $|Q[f(x)] - Q[A]| < \varepsilon$, $|\inf f(x) - \inf A| < \varepsilon$

From the relationship between the grey distance and the distance of the two real numbers, we know

$d(P[f(x)], P[A]) < \varepsilon$

$d(Q[f(x)], Q[A]) < \varepsilon$

$d(\inf f(x), \inf A) < \varepsilon$

therefore, $\lim_{x \to x_0} P[f(x)] = P[A]$, $\lim_{x \to x_0} Q[f(x)] = Q[A]$, $\lim_{x \to x_0} \inf f(x) = \inf A$.

Theorem 2 If $\lim_{x \to x_0} P[f(x)] = P[A]$, $\lim_{x \to x_0} Q[f(x)] = Q[A]$, $\lim_{x \to x_0} \inf f(x) = \inf A$,

then, $\lim_{x \to x_0} f(x) = A$

Proof:

Since $\lim_{x \to x_0} P[f(x)] = P[A]$, $\lim_{x \to x_0} Q[f(x)] = Q[A]$, $\lim_{x \to x_0} \inf f(x) = \inf A$, For any given real number $\varepsilon > 0$, there exists real numbers $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, so that, when $d(x, x_0) < \delta_1$, $d(x, x_0) < \delta_2$, $d(x, x_0) < \delta_3$,

$|P[f(x)] - P[A]| < \varepsilon$, $|Q[f(x)] - Q[A]| < \varepsilon$, and

$|\inf f(x) - \inf A| < \varepsilon$ will be tenable.

If we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and when $d(x, x_0) < \delta$, there must be

$$d(f(x), A) = \max\{|P[f(x)] - P[A]|, |Q[f(x)] - Q[A]|, |\inf f(x) - \inf A|\} < \varepsilon.$$

By definition: $\lim\limits_{x \to x_0} f(x) = A$.

Theorem 3 If $\lim\limits_{x \to x_0} f(x) = A$, and $\lim\limits_{x \to x_0} f(x) = B$, then $A = B$.

Proof: This is the uniqueness theorem of grey limit. It is proved by the uniqueness of the limit of multivariate funtions (here is three-variable), the key link being Theorem 1.

According to Theorem 1, $\lim\limits_{x \to x_0} P[f(x) = P[A]$, $\lim\limits_{x \to x_0} P[f(x)] = P[B]$ and $P[f(x)]$ is three-variable function of real value. Basing on the uniqueness of multivariate functions of real value, we have

$$P[A] = P[B].$$

In the same way,

$$Q[A] = Q[B], \quad \inf A = \inf B.$$

In other words, the two end points of both A and B are the same, and so are their inferior limits. Thus. A = B.

Reference

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(2) Wu Heqin ,Yue Changan ,Introduction of the grey mathematics ,Hebei people's publishing house ,1990.