INTEGRATION ON FUZZY MAPPING

MARIAN MATŁOKA

Department of Mathematics, Academy of Economics,
Al. Niepodległości 9, 60-967 Poznań, Poland

1. Introduction

The aim of the present paper is to explain new concept of an integral of fuzzy mapping. It should be clear that this integration is quite different from that of Sugeno [4], Klement [2], Kaleva [1]. We generalize the Riemann integral over a closed interval to fuzzy mappings.

2. Preliminaries.

Let $P(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P(R^n)$ as usual. Denote

$$E^n = \{ u : R^n \to \langle 0,1\rangle : u \text{ satisfies (i) - (iv) below} \},$$

where

(i) u is normal,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) supp u is compact.

For $0 < \alpha \leq 1$ let $u^{\alpha} = \{ x \in R^n : u(x) \geq \alpha \}$. Then from (i)-(iv) it follows that the α - level set $u^{\alpha} \in P(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function then according to the Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$g(u,v)(z) = \sup_{z=g(x,y)} \min (u(x),v(y)). \qquad (2.1)$$

It is well known that

$$[g(u,v)]^{\alpha} = g(u^{\alpha},v^{\alpha}) \qquad (2.2)$$

for all $u,v \in E^n$, $0 \leq \alpha \leq 1$ and g continuous (see [3]).

For the addition Eqn. (2.2) yields

$$[u + v]^{\alpha} = u^{\alpha} + v^{\alpha} \qquad (2.3)$$

Recall that the real numbers can be embedded to E^1 by the correspondence

$$c \rightarrow \bar{c}(t) = \begin{cases} 1 & \text{if } t=c, \\ 0 & \text{elsewhere.} \end{cases}$$

Aalagously to (2.1) we can also generalize the multiplication by a real number and for any real number c we obtain

$$[c \cdot u]^{\alpha} = c \cdot u^{\alpha} \qquad (2.4)$$

where 0 1 and u E^n.

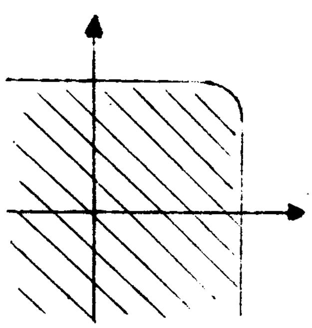
We metrize E^n by the metric

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d(u^{\alpha},v^{\alpha}),$$

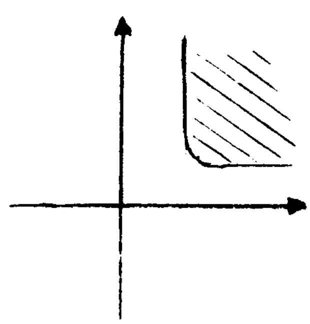
where d is the Hausdorff metric defined in $P(R^n)$.

Let us consider the sets $A,B \in P(R^n)$. The set A is called n-set (c-set) iff the following implication holds:

$$\text{if } x \in A \Rightarrow (-\infty, x\rangle \subset A \qquad (\langle x, \infty) \subset A).$$



n-set



c-set

We can extend the order relation, $\leq$, in $P(R^n)$ as follows:

$$A \leq B \text{ if and only if } nA \subset nB \text{ and } cA \supset cB,$$

where nA (cA) is the intersection of all n-sets (c-sets) containing the set A.

If $u, v \in E^n$ then the order relation between u and v we define in the following way:

$u \lesssim v$ if and only if for any $\alpha \in \langle 0,1 \rangle$ $u^{\alpha} \leq v^{\alpha}$.

If $u, v \in E^n$ then $u=v$ if and only if for any $x \in R^n$ $u(x)=v(x)$ or equivalently if for any $\alpha \in \langle 0,1 \rangle$ $u^{\alpha} = v^{\alpha}$.

3. Integral of fuzzy mapping

A finite set of numbers $t_0, t_1, \ldots, t_n$ is said to be a partition of the interval $\langle a,b \rangle$ if

$$a = t_0 \leq t_1 \leq \ldots \leq t_n = b.$$

We denote a partition of $\langle a,b \rangle$ by $P = \{ t_i : i=0,1,\ldots,n \}$, and define the mesh of P, written $|P|$, by

$$|P| = \max \{ t_j - t_{j-1} : j=1,\ldots,n \}.$$

Let F be a mapping from $\langle a,b \rangle$ to E^n such that for any $t \in \langle a,b \rangle$ $u \lesssim F(t) \lesssim v$. Such a mapping we will call a bounded fuzzy mapping. Define

$$m_i(F) = \inf \{ F(t) : t \in \langle t_{i-1}, t_i \rangle \}$$

$$M_i(F) = \sup \{ F(t) : t \in \langle t_{i-1}, t_i \rangle \},$$

where infimum and supremum are defined in sense of the order relation $\lesssim$.

The upper sum (U) and lower sum (L) for the mapping F and partition P are defined as follows

$$U(F,P) = \sum_{i=1}^{n} M_i(F) \cdot (t_i - t_{i-1})$$

and

$$L(F,P) = \sum_{i=1}^{n} m_i(F) \cdot (t_i - t_{i-1}),$$

where the sums and multiplications are defined according to the (2.3) and (2.4).

Define

$$\underline{\int_a^b} F = \sup\{ L(F,P) : P \in \mathcal{P} \}$$

and

$$\overline{\int_a^b} F = \inf\{ U(F,P) : P \in \mathcal{P} \},$$

where $\mathcal{P}$ is the set of all partitions of the interval $\langle a,b \rangle$.

A mapping F on $\langle a,b \rangle$ is said to be integrable on $\langle a,b \rangle$ if F is bounded on $\langle a,b \rangle$ and

$$\underline{\int_a^b} F = \overline{\int_a^b} F .$$

If F is integrable on $\langle a,b \rangle$ then the integral of F from a to b, written $\int_a^b F$, is defined by

$$\int_a^b F = \underline{\int_a^b} F = \overline{\int_a^b} F .$$

Theorem 3.1. If F is integrable on $\langle a,b \rangle$ and P is any partition of $\langle a,b \rangle$ then

$$L(F,P) \precsim \int_a^b F \precsim U(F,P) .$$

Proof. By the definition of $\underline{\int_a^b} F$ and $\overline{\int_a^b} F$ we have for any partition of $\langle a,b \rangle$

$$L(F,P) \precsim \underline{\int_a^b} F \quad \text{and} \quad \overline{\int_a^b} F \precsim U(F,P).$$

Now, F integrable on $\langle a,b \rangle$ implies $\underline{\int_a^b} F = \overline{\int_a^b} F$, and hence

$$L(F,P) \precsim \underline{\int_a^b} F = \int_a^b F = \overline{\int_a^b} F \precsim U(F,P).$$

Theorem 3.2. A bounded mapping F is integrable on $\langle a,b\rangle$ if and only if corresponding to each $\varepsilon > 0$ there is a partition P with the property that $D(U(F,P),L(F,P)) < \varepsilon$.

Proof. Assume that corresponding to each $\varepsilon > 0$ such an F exists. Then, since

$$D\left(\overline{\int_a^b} F , \underline{\int_a^b} F\right) \leq D(U(F,P),L(F,P)) < \varepsilon ,$$

it follows that $\underline{\int_a^b} F = \overline{\int_a^b} F$, and F is integrable on $\langle a,b\rangle$.

Conversely assume that F is integrable on $\langle a,b\rangle$. Then $\int_a^b F = \underline{\int_a^b} F = \overline{\int_a^b} F$. Therefore $\int_a^b F$ is the least upper bound of the sums $\{L(F,P)\}$ and is the greatest lower bound of $\{U(F,P)\}$. Hence corresponding to each $\varepsilon > 0$ there is a P_1 such that

$$D\left(\int_a^b F , L(F,P_1)\right) < \varepsilon/2$$

and a P_2 such that

$$D\left(U(F,P_2), \int_a^b F\right) < \varepsilon/2.$$

So,

$$D(U(F,P_2),L(F,P_1)) \leq D\left(U(F,P_2), \int_a^b F\right) +$$

$$+ D\left(L(F,P_1), \int_a^b F\right) < \varepsilon .$$

Let $P = P_1 \cup P_2$, P is a refinement of both P_1 and P_2. Therefore

$$D(U(F,P),L(F,P)) \leq D(U(F,P_2),L(F,P_1)) < \varepsilon .$$

This completes the proof.

References

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