Fuzzy Integrals of Set-valued Functions

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Abstract: In the paper, we extend fuzzy integrals of point-valued functions which was defined by Sugeno [5] to set-valued functions. Some properties similar to Auman integrals [1] are shown, and these include convexity, monotonicity, Fatou's lemma and Lebesgue convergence theorem, etc.

Keywords: fuzzy integral, set-valued function, set-valued fuzzy integral

1. Introduction

It is well known that set-valued functions have been used repeatedly in Economics [3]. Integrals of set-valued functions have been studied by Aumann [1], Debreu [2], and so on. But they are all based on classical Lebesgue integrals. Since Sugeno [5] brought out the concepts of fuzzy measure and fuzzy integrals, it has been extended and made deeper by Ralescu and Adams [6], Wang [1], Wang [8,9], Berres [11], etc. But the integrands are all point-valued functions. The paper's purpose is to define fuzzy integrals of set-valued functions (simply said to be set-valued fuzzy integrals). The method we will use here is just similar to Aumann's. Then it is an natural extension of fuzzy integrals of point-valued functions.

In the paper, the following concepts and notations will be used. Symbol I denotes the unite interval [0, 1], P(I) will denote the power set of I. The triplet $(X, \mathcal{A}, m)$ is a classical complete probability space (nonfuzzy). Let $\mu: \mathcal{A} \to I$ be a fuzzy measure of Sugeno's sense, and in addition, we assume μ satisfies the following conditions,

i) μ is null-additive, i.e. $\mu(A)=0$ implies $\mu(A \cup B)=\mu(B)$;

ii) $\mu \ll m$, i.e. $m(A)=0$ implies $\mu(A)=0$.

The fuzzy integral of a measurable point-valued function $f: X \to I$ is defined as

$$\int_A f d\mu = \bigvee_{\alpha \in I} (\alpha \wedge \mu(A \cap F_\alpha))$$

where $F_\alpha = \{x \in X: f(x) \geq \alpha\}$, $A \in \mathcal{A}$.

Obviously, for the fuzzy measure μ satisfied the above conditions, we have:

If $f_1(x)=f_2(x)$ for $x \in X$ m-a.e, then

$$\int f_1 \, d\mu = \int f_2 \, d\mu$$

A set-valued function is a mapping F from X to $P(I) \setminus \{\phi\}$. By a measurable set-valued function F, it means that its graph is measurable, i.e.

$$GrF = \{(x, r) \in X \times I: \; r \in F(x)\} \in \mathcal{A} \times B(I)$$

where B(I) is the Borel field of I.

The rest of the paper consists of two parts. In section 2, we will define set-valued fuzzy integrals at first. Then its properties will be shown. In section 3, our main task is the discussion of convergence of sequence of set-valued fuzzy integrals. We will give Fatou's lemma and Lebesgue convergence theorem.

2. Definition and propositions.

In this section, by the similar way of Aumann's set-valued integrals, we first define set-valued fuzzy integrals, then we study it in detail. Let's start with the definition.

Definition. Let F be a set-valued function, $A \in \mathcal{A}$. Then the fuzzy integral of F on A is defined as

$$\int F d\mu = \{ \int f d\mu ,\ f \in S(F) \}$$

where $S(F) = \{ f \text{ is measurable}, f(x) \in F(X), x \in X \text{ m-a} \cdot \text{e} \}$.
i · e. the family of measurable selectionsof F.

Instead of $\int_X F d\mu$, we will write $\int F d\mu$. Obviously, $\int F d\mu$, may be empty.

A set-valued function F is said to be integrable if $\int F d\mu \neq \phi$.

Next, we will give a sufficient condition for which F is integrable.

Proposition 2.1 If F is a measurable set-valued function, then $\int F d\mu \neq \phi$.

Proof. By the measurablity of F, there exist a m-a · e measurable selection of F. This implies $\int F d\mu \neq \phi$. (Q.E.D)

From this proposition, we can see that Sugeno's fuzzy integrals is as pecial case of the set-valued fuzzy integrals. Since a point-valued function can be view as a special set-valued function.

Remark. With respect to an arbitrary fuzzy measure (no any restriction) , we can also define the set-valued fuzzy integral by the same way.

A set-valued function F is said to be convex-valued, if F(x) is convex for every $x \in X$. The following proposition will show that the fuzzy integral of convex-valued set-valued functions is convex.

Proposition 2. 2 If a measurable set-valued function F is convex-valued, then $\int F d\mu$ is convex.

Proof. If $\int F d\mu$ is a single point set, then it is convex. Otherwise, let $y_1, y_2 \in \int F d\mu$, and $y_1 < y_2$. Then there exist $f_1, f_2 \in S(F)$, s.t.

$$y_1 = \int f_1 d\mu, \qquad y_2 = \int f_2 d\mu,$$

Further, let $y \in (y_1, y_2)$, we need to find a $f \in S(F)$, s.t. $y = \int f d\mu$. Define

$$f(x) = \begin{cases} y, & x \in (f_1 \leqslant y \leqslant f_2) \\ f_1 & x \in ((f_1 \wedge f_2) > y) \cup (f_1 > y > f_2) \\ f_2 & x \in ((f_1 \vee f_2) < y) \end{cases}$$

where $(f_1 \leqslant y \leqslant f_2)$ denotes the set $\{x \in X, f_1(x) \leqslant y \leqslant f_2(x)\}$, and the rest is similar.

It is easy to see that $f \in S(F)$, and furthermore

$\mu(f \geqslant y) \geqslant \mu(f_2 \geqslant y) \geqslant \mu(f_2 \geqslant y_2) \geqslant y_2 \geqslant y$

$\mu(f > y) = \mu(f_1 > y) \leqslant \mu(f_1 > y_1) \leqslant y_1 < y$

Then we obtain $y = \int f d\mu$. By the arbitrary of y, and y_1, y_2, we know that $\int F d\mu$ is an interval. (Q.E.D)

Further, Let F be a set-valued function, coF(x) will denote the convex hull of F(x) for each $x \in X$. Then there holds the following .

Proposition 2.3 If F is a measurable set-valued function, then

$$co \int F d\mu = \int coF d\mu$$

Proof. The relation "$\subset$" is obvious. To obtain the converse relation "$\supset$" , we assume $y \in \int coF d\mu$. Then there exists $f \in S(coF)$, s.t. $y = \int f d\mu$. If f is in S(F), then $y \in co \int F d\mu$. Otherwise, we assume y is not in S(F). Since by setting

$$GrE = GrF \cap \{(x, r) \in X \times I, f(x) \leqslant r \leqslant 1\}$$

is measurable, nonempty, then there exists a m-a.e measurable selection f_1 of E, further $f_1 \in S(F)$, and

$$\int f d\mu \leq \int f_1 d\mu .$$

Similarly, we can find $f_2 \in S(F)$, s. t.

$$\int f d\mu \geq \int f_2 d\mu$$

Consequently, we have proved $y \in co\int F d\mu$. (Q.E.D)

The set-valued function F is said to be closed-valued if $F(x)$ is closed for every $x \in X$.

Proposition 2. 4 If a measurable set-valued function F is closed-valued, then $\int F d\mu$ is closed.

Proof . It is easy to see that S(F) is a closed subset of the space I^I (with the usual product topology). Then S(F) is compact. Since fuzzy integral "$\int \cdot d\mu$" is continuous with the view of function, then $\int F d\mu$ is compact. Therefore , it is closed.(Q.E.D)

Corollary. Let F be a measurable interval-valued function. i.e. $F(x) = [f^-(x), f^+(x)]$. Then

(i) $f^-, f^+ \in S(F)$;

(ii) $\int F d\mu = [\int f^- d\mu, \int f^+ d\mu]$

Proof. Since F is closed-valued and measurable, by Castaing representation, then there exist $\{f_n\} \subset S(F)$, s. t.

$$F(x) = cl\{f_n(x)\} \qquad \text{(cl is closed hull)}$$

Then $f^-(x) = \inf\{f_n(x)\}$, $f^+(x) = \sup\{f_n(x)\}$, Consequently, f^-, f^+ are measurable, (i) is proved, (ii) is a direct result of proposition 2.2 and 2.4. (Q.E.D).

The above properties are all related to "convexity" . We will deal with "monotonicity" in the following.

Let $A, B \in P(I)$, by $A \leqslant B$ means that,

(i) For each $x_0 \in A$, there exists $y_0 \in B$, s.t. $x_0 \leqslant y_0$;

(ii) For each $y_0 \in B$, there exists $x_0 \in A$, s.t. $x_0 \leqslant y_0$.

Proposition 2.5 Let F be a measurable set-valued function, $A, B \in \mathcal{A}$ Then $A \subset B$ implies

$$\int_A F d\mu \leqslant \int_B F d\mu$$

Proposition 2.6 Let F_1 and F_2 be two measurable set-valued function. Then $F_1 \leqslant F_2$ implies

$$\int F_1 d\mu \leqslant \int F_2 d\mu .$$

The proof of Proposition 2.5 is direct, and the proof of Proposition 2.6 is similar to the proof of proposition 2.3, here we omit it.

3. Convergence theorem.

In this section, Fatou's lemma, Lebesgue convergence theorem will be shown. We begin with the concept of convergence of sequence of elements in P(I).

Let $\{An\} \subset P(I)$ be a sequence, we define

$$\text{Limsup } An = \{x;\ x = \lim_{k \to \infty} x_{n_k},\ x_n \in An\ (n \geqslant 1)\}$$

$$\text{Liminf } An = \{x;\ x = \text{Lim } x_n,\ x_n \in An\ (n \geqslant 1)\}$$

If Limsup An = Liminf An = A, we say that {An} is convergent to A, and it is simply written by Lim An = A or $An \to A$.

Theorem 3.1 (Fatou's lemma) If $\{F_n\}$ is a sequence of measurable set-valued functions, then

$$\text{Limsup} \int F_n d\mu \subset \int \text{Limsup } F_n d\mu .$$

Proof , Let $y \in \text{Limsup} \int F_n d\mu$. Then from the definition of limit superior we have that $y = \lim_{k\to\infty} y_{n_k}$, and $y_n \in \int F_n d\mu$ $(n \geq 1)$. So $y_{n_k} = \int f_{n_k} d\mu \to y$ $(k \to \infty)$ where $f_{n_k} \in S(F_{n_k})$. Since $\{f_{n_k}\}$ is included in the compact space I^X (with the usual product topology), we can find a subsequence $\{f_{n_m}\}$ of $\{f_{n_k}\}$, and $\{f_{n_m}\}$ is convergent. So $\int f_{n_m} d\mu \to y'$ $(m \to \infty)$. Since the limits are unique, therefore we have that $y = \int f_{n_m} d\mu$, and this implies that $y \in \int \text{Limsup} F_n d\mu$. (Q.E.D)

Theorem 3.2 If $\{F_n\}$ is a sequence of measurable set-valued functions, then

$$\int \text{Liminf} F_n d\mu \subset \text{Liminf} \int F_n d\mu .$$

Proof . Let $y \in \int \text{Liminf} F_n d\mu$. Then $y = \int f d\mu$. where $f \in S(\text{Liminf} F_n)$. Write $I^\infty = I \times I \times \cdots$, then I^∞ is a complete metric space (with the metric induced by the usual product topology). For each $x \in X$, define a subset $G(x)$ of I^∞ by

$$G(x) = \{(y_1, y_2, \cdots);\ y_n \in F_n(x)\ (n \geq 1),\ \text{and } \text{Lim } y_n = f(x)\}$$

Then the statement $f \in S(\text{Liminf } F_n)$ is equivalent to the statement that G is a set-valued mapping. Furthermore, we can easily prove that G is measurable (for a metric space the measurablity is same defined). Then there exists a m-a.e measurable selection g of G, that is a sequence of measurable function $\{f_n\}$, s.t. $f_n \in S(F_n)(n \geq 1)$, and $\text{Lim } f_n(x) = f(x)$. This implies $\int f_n d\mu \to \int f d\mu = y$. Consequently $y \in \text{Liminf} \int F_n d\mu$.
(Q.E.D)

From the above two theorems, we can easily obtain the following Lebesgue convergence theorem.

Theorem 3.3 If $F_n(x) \to F(x)$ for $x \in X$ m-a.e. and all the F_n are measurable, then $\int F_n d\mu \to \int F d\mu$.

Proof, since $F(x) = \text{Liminf } F_n(x) = \text{Limsup } F_n(x)$, then by theorem 3.1 and 3.2.

$$\int F d\mu = \int \text{Liminf } F_n d\mu \subset \text{Liminf} \int F_n d\mu \subset \text{Limsup} \int F_n d\mu \subset \int \text{Limsup } F_n d\mu = \int F d\mu.$$

Hence, $\text{Lim} \int F_n d\mu$ exists and equals to $\int F d\mu$. (Q.E.D)

Remark.

(i) We can also prove the Corollary in section 2 by using theorem 3.1.

That is , if we set $F_n = F$, then $\text{Limsup} F_n = F$. So

$$\text{Limsup} \int F_n d\mu \subset \int \text{Limsup} F_n d\mu = \int F d\mu$$

This implies $\int F d\mu$ is closed.

(ii) Throughout the paper, our set-valued function is just valued in P(I). Similarly, we can deal with the case for which the set-valued function is valued in $P(R^+)$. These will be shown in our another paper.

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