A NOTE TO THE RADON-NIKODÝM THEOREM FOR FUZZY MEASURE SPACES

Radko MESIAR
Department of Mathematics, Slovac Technical University,
Radlinského 11, 813 68 Bratislava, Czechoslovakia

Dvurečenskij in [1] have proved the Radon-Nikodým theorem for fuzzy probability spaces. In the present paper, we give a simplier proof of this theorem using the fuzzified random variables presented in [8]. Our method allows to present many other interesting results of probability theory.

Keywords: Fuzzy measure space, Fuzzy observable, Fuzzified random variable, the Radon-Nikodým theorem

1. Introduction

Let (Ω, M) be a fuzzy quantum space in sense of [3], i.e. Ω is a nonempty set, $M \subset [0,1]^{\Omega}$ is a soft fuzzy σ-algebra of fuzzy subsets of Ω (see [9]). Recall that it means

$$0_{\Omega} \in M \tag{1.1}$$

$$a \in M \Longrightarrow a' = 1 - a \in M \tag{1.2}$$

$$\{a_i\} \subset M \Longrightarrow \bigvee a_i = \sup a_i \in M \tag{1.3}$$

$$(1/2)_{\Omega} \notin M \tag{1.4}$$

A fuzzy observable x of (Ω, M) (see [3]) is a σ-homomorphism, $x : \mathcal{B} \longrightarrow M$ ($\mathcal{B}$ is the system of all Borel subsets of real line),

$$\forall E \in \mathcal{B} : x(E^c) = x(E)' \tag{1.5}$$

$$\forall \{E_i\} \subset \mathcal{B} : x(\bigcup E_i) = \bigvee x(E_i) \tag{1.6}$$

We recall some notions and results of [1].

Let $a \in M$ be a fuzzy subset of Ω. Let define a mapping x_a, $x_a : \mathcal{B} \longrightarrow M$, via

$$x_a(E) = \begin{cases} a \wedge a' & \text{if } 0 \text{ and } 1 \notin E \\ a' & \text{if } 0 \in E \text{ and } 1 \notin E \\ a & \text{if } 0 \notin E \text{ and } 1 \in E \\ a \vee a' & \text{if } 0 \text{ and } 1 \in E \end{cases} \tag{1.7}$$

Then x_a is a fuzzy observable of (Ω, M) called an indicator of a fuzzy set a.

A mapping $v: M \longrightarrow R$ is said to be a signed fuzzy measure (see [1]), iff

$$\forall a \in M: \; v(a \vee a') = v(1_\Omega) \tag{1.8}$$

$$\forall \{a_i\} \subset M, \; a_i \leq a_j' \text{ whenever } i \neq j: \; v(\vee a_i) = \sum v(a_i) \tag{1.9}$$

If v is positive, then it is named a fuzzy measure. If for a measure m we have $m(1_\Omega) = 1$, then m is a fuzzy P-measure (see also [9]).

We say that a signed measure v is dominated by a measure m, $v \ll m$, if $m(a) = 0$ implies $v(a) = 0$.

The measure spectrum of an observable x in a measure m is the set

$$\sigma_m(x) = \bigcap \{C: C \text{ is closed, } m(x(C)) = m(1_\Omega)\} \tag{1.10}$$

Let m be a fuzzy measure on (Ω, M). A mapping m_x , $m_x(E) = m(x(E))$, $E \in \mathcal{B}$, is a measure on $\mathcal{B}$. This enables to define an integral of x with respect to a fuzzy measure m by the expression

$$\int x \, dm = \int_R t \, dm_x(t) \tag{1.11}$$

if the right-hand side exists and is finite. If h is a Borel-transformation, then $h(x(E)) = x(h^{-1}(E))$ defines a fuzzy observable of (Ω, M) and under the conventional assumptions on the integrability we have

$$\int h(x) \, dm = \int_R h(t) \, dm_x(t) \tag{1.12}$$

In [3,4] was introduced the algebraic structure of the space of fuzzy observables. Note, that $z = x + y$ iff

$$\forall t \in R: \ z(]-\infty,t[) = \bigvee_{r \in Q} \left(x(]-\infty,r[) \wedge y(]-\infty,t-r[) \right) \quad (1.13)$$

Here Q is the set of all rational numbers.

We say that two fuzzy observables x and y of (Ω, M) are equal almost everywhere with respect to a fuzzy measure m, $x = y$ a.e.[m], if $\sigma_m(x - y) = \{0\}$.

We define an indefinite integral of an observable x over a fuzzy set $a \in M$ via

$$v(a) = \int_a x \, dm = \int x . x_a \, dm \quad (1.14)$$

It is easy to see that v defined in (1.14) is a signed fuzzy measure dominated by m. More, if $x = y$ a.e.[m], then for every $a \in M$ we have $\int_a x \, dm = \int_a y \, dm$. A converse assertion is the mean result of [1].

Theorem 1.1.(Radon-Nikodým). Let v be a signed fuzzy measure on a fuzzy quantum space (Ω, M), which is dominated by a measure m. Then there exists a fuzzy observable x of (Ω, M) such that

$$\forall a \in M: \ v(a) = \int_a x \, dm .$$

If $v(a) = \int_a y \, dm$, for every $a \in M$, for another fuzzy observable y, then $x = y$ a.e.[m].

In [8], we have introduced a notion of a fuzzified real random variable (f,c), where f is a real random variable and c some fuzzy subset of M. We have given an algebraic structure preserving isomorphism of fuzzy observables and classes of equivalence of fuzzified random variables. For details see section 2. The calculus of fuzzified random variables is simplier and clearer. This fact simplifies the situation and gives a new tool for proving the results of the fuzzy observable theory.

2. Fuzzified random variables

Piasecki in [9] had shown that a fuzzy probability space (Ω, M, m), where m is a fuzzy P-measure on M, induces a classical probability space $(\Omega, K(M), P_m)$. Here

$$K(M) = \{A \subset \Omega : \exists\, a \in M, \{a > 1/2\} \subset A \subset \{a \geq 1/2\}\} \tag{2.1}$$

$P_m(A) = m(a)$ for any $A \in K(M)$, where the relation of a and A is given by (2.1) (2.2)

Definition 2.1. Let f be a K(M)-measurable real random variable. Then a couple (f,c), $c \in M$, $c \geq c'$, will be called a fuzzified random variable on (Ω, M) iff for every $E \in \mathcal{B}$ we have

$x(E) \in M$, where

$$x(E) = \begin{cases} c & \text{on } f^{-1}(E) \\ c' & \text{else} \end{cases} \tag{2.3}$$

It is easy to see that x defined by (2.3) is a fuzzy observable of (Ω, M). On the other hand, let x be any fuzzy observable of (Ω, M). Then for any fixed $\omega \in \Omega$ we have

$$\forall\, t \in R:\ x(\omega,t) = x(]-\infty,t[)(\omega) = \begin{cases} x(R)(\omega) & \text{if } t > f_x(\omega) \\ x(R)'(\omega) & \text{if } t \leq f_x(\omega) \end{cases} \tag{2.4}$$

For details see e.g. [5,7]. Here $f_x(\omega)$ is a real constant uniquelly determined by x and ω if $x(R)(\omega) > 1/2$. If $x(R)(\omega)$ is equal to 1/2, then $x(\omega,t) = 1/2$ for any t, so that $f_x(\omega)$ can be chosen arbitrarily. The function f_x will be called a crisp projection of x and it is a K(M)-measurable real random variable, see [5]. It is easy to see that the couple $(f_x, x(R))$

is a fuzzified random variable. Further, if g_x is another crisp projection of x, then

$$\{f_x \neq g_x\} \subset \{x(R) = 1/2\} \tag{2.5}$$

The results of [2,6] show that, for any random variable f on $(\Omega, K(M))$, there exists a fuzzified random variable (f,c). Two fuzzified random variables (f,c) and (g,d) are said to be equivalent if they induce the same fuzzy observable x, i.e. if $c = d$ and $\{f \neq g\} \subset \{c = 1/2\}$.

In[8] is defined an algebraic structure of fuzzified random variables.

$$(f,c) \oplus (g,d) = (f+g, c \wedge d) \tag{2.6}$$

$$(f,c) \odot (g,d) = (f.g, c \wedge d) \tag{2.7}$$

If h is a Borel transformation (1- or 2-dimensional), then

$$\begin{aligned} &h((f,c)) = (h(f),c) \text{ , or} \\ &h((f,c),(g,d)) = (h(f,g), c \wedge d) \end{aligned} \tag{2.8}$$

is a fuzzified random variable.

The main result of [8] is the next one.

Theorem 2.1. (2.3) and (2.4) define an algebraic structure preserving isomorphism of fuzzy observables of (Ω, M) and classes of equivalence of fuzzified random variables on (Ω, M).

Example 2.1. Let x_a be an indicator of a fuzzy set $a \in M$. Then the corresponding class of fuzzified random variables is $\{(f, a \vee a'), f = 1 \text{ if } a > a', f = 0 \text{ if } a < a'\}$. Let I_A be a characteristic function of a crisp set A. Then $A \in K(M)$, $\{a > 1/2\} \subset A \subset \{a \geq 1/2\}$ iff I_A is a crisp projection of x_a.

3. The Radon-Nikodým theorem

At first, we introduce some simple facts.

Lemma 3.1. Let v be a signed fuzzy measure on (Ω, M). Then the mapping $P_v: K(M) \longrightarrow R$,

$$P_v(A) = v(a) \quad \text{for } a, A \text{ given by (2.1)} \tag{3.1}$$

is a signed measure on $K(M)$. If v is a fuzzy measure, then P_v is a measure. If v is a fuzzy P-measure, then P_v is a probability measure.

Proof. For the last case of a fuzzy P-measure see [9]. The other cases are similar.

Lemma 3.2. Let x and y be two fuzzy observables of (Ω, M) and let f_x and f_y be the corresponding crisp projections. Then $x = y$ a.e.[m] iff $f_x = f_y$ a.e.$[P_m]$ for any fuzzy measure m defined on (Ω, M).

Proof. We have $\sigma_m(z) = \{0\}$ iff $m(z\{0\}) = P_m(f_z^{-1}(\{0\})) = 1$.
Next, $f_x - f_y$ is a crisp projection of $x - y$. Now, it is enough to take into account the fact that $x = y$ a.e.[m] iff $\sigma_m(x-y)$ is $\{0\}$.

Lemma 3.3. Let x be a fuzzy observable of (Ω, M) and let f_x be its crisp projection. Let $a \in M$ and $A \in K(M)$ be given (in sense of (2.1)). Then for any fuzzy measure m on M we have

$$\int_a x \, dm = \int_A f_x \, dP_m \tag{3.2}$$

Proof. The fact $m(x(E)) = P_m(f_x^{-1}(E))$, for any $E \in \mathcal{B}$, implies $\int x \, dm = \int f_x \, dP_m$. On the other hand, I_A is a crisp projection of x_a, hence $f_x . I_A$ is a crisp projection of $x . x_a$. It follows $\int_a x \, dm = \int x . x_a \, dm = \int f_x . I_A \, dP_m = \int_A f_x \, dP_m$ q.e.d.

Proof of the Theorem 1.1.

i) Let v be a signed fuzzy measure on (Ω, M) dominated by a fuzzy measure m. Then the signed measure P_v on $(\Omega, K(M))$ is dominated by the measure P_m. The classical Radon-Nikodým theorem implies the existence of a $K(M)$-measurable integrable random variable f such that

$$\forall A \in K(M): \quad P_v(A) = \int_A f \, dP_m \quad .$$

Then there exists a fuzzy observable x (see e.g. [2,6]) of (Ω, M) such that f is its crisp projection. Now, Lemma 3.3. implies

$$\forall a \in M: \quad v(a) = \int_a x \, dm$$

ii) Let y be another fuzzy observable of (Ω, M) such that for any $a \in M$ we have $v(a) = \int_a y \, dm$. Let g be a crisp projection of y. Then for any $A \in K(M)$ we get $\int_A f \, dP_m = \int_A g \, dP_m$. It follows $f = g$ a.e. $[P_m]$ and Lemma 3.2. implies $x = y$ a.e. $[m]$ q.e.d.

Remark 3.1. A crisp projection f_x of a fuzzy observable x is in fact a representation of x by a real random variable presented in [2] . This fact imply the similar ideas of our paper and of [2] in proving Theorem 1.1. Note that the methods used in [2] are mainly based on the Loomis-Sikorski theorem. On the other side, our method is based on the isomorphism of fuzzy observables and classes of equivalence of fuzzified random variables. This isomorphism is a constructive one due to (2.3) and (2.4).

4. Applications

Let (Ω, M, m) be a fuzzy probability space, i.e. m is a fuzzy P-measure on M. Let M_o be a soft fuzzy sub-σ-algebra of M. For a given fuzzy observable x of (Ω, M), we call a fuzzy observable y of (Ω, M_o) as the conditional mean value of x with respect to M_o, $y = E_m(x/M_o)$, if for any $a \in M_o$ we have

$$\int_a y \, dm = \int_a x \, dm \qquad (4.1)$$

The existence of $E_m(x/M_o)$ is shown in [1]. It is easy to see that for a crisp projection f_y of y we have

$$f_y = E_{P_m}(f_x/K(M_o)) \qquad (4.2)$$

It means that a crisp projection of a conditional mean value of a fuzzy observable x is a conditional mean value of its crisp projection f_x.

If z is another fuzzy observable of (Ω, M), denote

$$M_z = \{z(E),\ E \in \mathcal{B}\} \cup \{0_\Omega, 1_\Omega\} \qquad (4.3)$$

Then by [1] we can define a conditional mean value of x with respect to z as follows:

$$y = E_m(x/z) = E_m(x/M_z) \qquad (4.4)$$

It is easy to see that y is isomorphic to a class of fuzzified random variables $\{(f_y, z(R))\}$, where $f_y = E_{P_m}(f_x/f_z)$. By (2.3) we get

$$\forall E \in \mathcal{B} : y(E) = \begin{cases} z(R) & \text{on } f_y^{-1}(E) \\ z(R)' & \text{else} \end{cases} \qquad (4.5)$$

For a conditional probability $P_m(a/b)$, a, $b \in M$, defined in [1], we get

$$\forall E \in \mathcal{B} : P_m(a/b) = \begin{cases} b \wedge b' & \text{if } m(a/b) \notin E,\ m(a/b') \notin E \\ b & \text{if } m(a/b) \in E,\ m(a/b') \notin E \\ b' & \text{if } m(a/b) \notin E,\ m(a/b') \in E \\ b \vee b' & \text{if } m(a/b) \in E,\ m(a/b') \in E \end{cases} \qquad (4.6)$$

Note that by Piasecki [9] we have

$$m(a/b) = m(a \wedge b)/m(b) \quad \text{if } m(b) > 0 \qquad (4.7)$$

For $m(b) = 0$ we can choose $m(a/b)$ arbitrarily, e.g. $m(a/b) = 0$.

In the classical probability theory we have

$$P(I_A/I_B) = h(I_B) \qquad (4.8)$$

where h is any injective Borel transformation such that

$h(0) = P(A/B')$ and $h(1) = P(A/B)$. In the fuzzy case we are in quite similar situation. (4.6) implies

$$P_m(a/b) = h(x_b) \tag{4.9}$$

where h is any injective Borel transformation such that $h(0) = m(a/b')$ and $h(1) = m(a/b)$.

Using formula (4.2) we can prove many other interesting results, e.g. the martingale convergence theorem. From other possible results, we can introduce the strong law of large numbers, the central limit theorem, the Hahn-Jordan decomposition of fuzzy measures, more-dimensional fuzzy observables calculus etc.

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