The uniqueness of the extension of fuzzy measure

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Abstract

In this paper, it is proved that if a Sugeno's fuzzy measure on algebra $\mathcal{A}_0$ can be extended onto the σ- algebra $\sigma(\mathcal{A}_0)$ generated by $\mathcal{A}_0$, then the extension is unique.

1. Introduction

The extension problems of Sugeno's fuzzy measure have been discussed by many researchers. Many extension theorems are obtained. In this paper, unlike other researchers, we shall not pay attention to the conditions which a fuzzy measure on algebra $\mathcal{A}_0$ can be extended onto σ- algebra $\sigma(\mathcal{A}_0)$, we shall prove the uniqueness of the extension if the extension is possible.

Let X be the universe, $\mathcal{P}(X) = \{A;\ A \text{ is a set of } X\}$ and $\sigma(\mathcal{A}_0)$ be the σ- algebra generated by $\mathcal{A}_0 \subset \mathcal{P}(X)$.

2. Fuzzy measures on algebra and on σ- algebra

Definition 2.1 Let $\mathcal{A}_0 \subset \mathcal{P}(X)$ be an algebra. Set function $u: \mathcal{A}_0 \to [0, 1]$ is called a fuzzy measure on $\mathcal{A}_0$, if u satisfies

(1) $u(\emptyset) = 0$, $u(X) = 1$;

(2) $\forall A, B \in \mathcal{A}_0$, $A \subset B \Rightarrow u(A) \leq u(B)$;

(3) $\forall A_n \in \mathcal{A}_0$, $n = 1, 2, \ldots$, $A \in \mathcal{A}_0$, $A_n \nearrow A \Rightarrow \lim_n u(A_n) = u(A)$;

(4) $\forall A_n \in \mathcal{A}_0$, $n = 1, 2, \ldots$, $A \in \mathcal{A}_0$, $A_n \searrow A \Rightarrow \lim_n u(A_n) = u(A)$.

Definition 2.2 Let $\sigma \subset \mathcal{P}(X)$ be a σ- algebra. Set function $u: \sigma \to [0, 1]$ is called a fuzzy measure on σ if u satisfies

(1) $u(\emptyset) = 0$, $u(X) = 1$;

(2) $\forall A, B \in \sigma$, $A \subset B \Rightarrow u(A) \leq u(B)$;

(3) $\forall A_n \in \sigma$, $n = 1, 2, \ldots$, $A_n \nearrow \Rightarrow u(\lim_n A_n) = \lim_n u(A_n)$;

(4) $\forall A_n \in \sigma$, $n = 1, 2, \ldots$, $A_n \searrow \Rightarrow u(\lim_n A_n) = \lim_n u(A_n)$.

3. The extension of fuzzy measure

<u>Definition</u> 3.1 Let u be a fuzzy measure on algebra $\mathcal{A}_0 \subset \mathcal{P}(X)$, $\sigma \subset \mathcal{P}(X)$ be a σ- algebra such that $\mathcal{A}_0 \subset \sigma$, $\mathcal{A}_0 \neq \sigma$. If there exists set function $\tilde{u}: \sigma \to [0, 1]$ which satisfies (1) $\tilde{u}$ is a fuzzy measure on σ, (2) $\tilde{u}|\mathcal{A}_0 = u$, then $\tilde{u}$ is called an extension of u from $\mathcal{A}_0$ to σ, where $\tilde{u}|\mathcal{A}_0$ is the restriction of $\tilde{u}$ to $\mathcal{A}_0$.

<u>Definition</u> 3.2 Let u be a fuzzy measure on algebra $\mathcal{A}_0$. If for arbitary two extensions of u from $\mathcal{A}_0$ to σ- algebra σ which contains $\mathcal{A}_0$, u_1, u_2,

$$u_1 = u_2$$

we call the extension of u from $\mathcal{A}_0$ to σ unique.

4. The construction of σ- algebra $\sigma(\mathcal{A}_0)$

The idea of this construction of $\sigma(\mathcal{A}_0)$ come from [6].

Suppose $\mathcal{A}_0$ is an algebra. Write

$$\mathcal{A}_1 = \{A;\ \exists A_n \in \mathcal{A}_0,\ A_n \uparrow A\};$$
$$\mathcal{A}_2 = \{A;\ \exists A_n \in \mathcal{A}_1,\ A_n \downarrow A\};$$
$$\mathcal{A}_3 = \{A;\ \exists A_n \in \mathcal{A}_2,\ A_n \uparrow A\};$$
$$\mathcal{A}_4 = \{A;\ \exists A_n \in \mathcal{A}_3,\ A_n \downarrow A\};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathcal{A}_w = \bigcup_{v<w} \mathcal{A}_v; \quad \text{(w is the least infinite ordinal number)}$$
$$\mathcal{A}_{w+1} = \{A;\ \exists A_n \in \mathcal{A}_w,\ A_n \uparrow A\};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathcal{A}_{2w} = \bigcup_{v<2w} \mathcal{A}_v;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

<u>Theorem</u> 4.1 The set classes defined above satisy that $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_w \subset \mathcal{A}_{w+1} \subset \cdots \subset \mathcal{A}_v \subset \cdots \subset \sigma(\mathcal{A}_0)$ (v may not be countable ordinal numbers) and there exists ordinal number r such that $\mathcal{A}_r = \sigma(\mathcal{A}_0)$.

Proof (1) for each v, A, B $\in \mathcal{A}_v \Rightarrow A - B \in \mathcal{A}_{v+2}$;

(2) For each v, if $A_n \in \mathcal{A}_v$, n = 1, 2, ..., $A_n \uparrow$ or $\downarrow$), then $\lim_n A_n \in \mathcal{A}_{v+2}$.

By using (1), (2) and infinite induction, we can prove Theorem4.1.

5. Consistent concept of set function

Definition 5.1 Let $\mathcal{C} \subset \mathcal{P}(X)$. Set function $u: \mathcal{C} \to [0, 1]$ is called lower(resp. upper) consistent on $\mathcal{C}$, if $\forall B \in \mathcal{C}$, $\forall A_n \in \mathcal{C}$, $n = 1, 2, \ldots$, $A_n \uparrow \bigcup_{n=1}^{\infty} A_n \supset B \Rightarrow \lim_n u(A_n) \geqq u(B)$ (resp. $A_n \searrow \bigcap_{n=1}^{\infty} A_n \subset B \Rightarrow \lim_n u(A_n) \leqq u(B)$). u is called consistent on $\mathcal{C}$, if u is both lower and upper consistent on $\mathcal{C}$.

6. The uniqueness of the extension of fuzzy measure

In this section, we shall prove the main result of this paper.

Theorem 6.1 Suppose u is a fuzzy measure on algebra $\mathcal{A}_0$. In the condition that u can be extended to a fuzzy measure $\tilde{u}$ on $\sigma(\mathcal{A}_0)$(we simply call this condition (*) condition), we have the extension is unique.

Lemma 6.1 In (*) condition, we have u is consistent on $\mathcal{A}_0$.

From Lemma6.1, we can verify

Lemma 6.2 $\forall A_n, B_n \in \mathcal{A}_0$, $n = 1, 2, \ldots$, $A_n \uparrow A$, $B_n \uparrow A \Rightarrow \lim_n u(A_n) = \lim_n u(B_n)$.

Definition 6.1 For $A \in \mathcal{A}_1$, define

$$u_1(A) = \lim_n u(A_n)$$

where $A_n \in \mathcal{A}_0$, $n = 1, 2, \ldots$, $A_n \uparrow A$.

Lemma6.2 makes Definition6.1 rational.

Lemma 6.3 In (*) condition, we have $\tilde{u}\,|_{\mathcal{A}_1} = u_1$ and u_1 is consistent on $\mathcal{A}_1$.

Lemma6.4 follows from Lemma6.3.

Lemma 6.4 $\forall A_n, B_n \in \mathcal{A}_1$, $n = 1, 2, \ldots$, $A_n \searrow A$, $B_n \searrow A \Rightarrow \lim_n u_1(A_n) = \lim_n u_1(B_n)$.

Definition 6.2 For $A \in \mathcal{A}_2$, define

$$u_2(A) = \lim_n u_1(A_n)$$

where $A_n \in \mathcal{A}_1$, $n = 1, 2, \ldots$, $A_n \searrow A$.

Lemma6.4 makes Definition6.2 rational.

Lemma 6.5 In (*) condition, we have $\tilde{u}\,|_{\mathcal{A}_2} = u_2$ and u_2 is consistent on $\mathcal{A}_2$.

Similarily, we can uniquely define $u_3, u_4, \ldots, u_v, \ldots$ on $\mathcal{A}_3, \mathcal{A}_4, \ldots, \mathcal{A}_v, \ldots (v < w)$ and obtain $\tilde{u}\,|_{\mathcal{A}_v} = u_v (v < w)$.

Definition 6.3 For $A \in \mathcal{A}_w$, define

$$u_w(A) = u_v(A)$$

where $A \in \mathcal{A}_v (v < w)$.

Lemma 6.6 In (*) condition, we have $\tilde{u}|_{\mathcal{A}_w} = u_w$ and u_w is consistent on $\mathcal{A}_w$.

Regarding u_w and $\mathcal{A}_w$ as u and $\mathcal{A}_0$ respectively, we can repeat above process and obtain that in (*) condition, $\tilde{u}|_{\mathcal{A}_v} = u_v$, $v < 2w$.

By using infinite induction, we can verify that in (*) condition, $\tilde{u}|_{\mathcal{A}_v} = u_v$ for all $\mathcal{A}_v$.

Let $v = r$, we have $\mathcal{A}_r = \sigma(\mathcal{A}_0)$ and

$$\tilde{u} = \tilde{u}|_{\mathcal{A}_r} = u_r$$

and Theorem6.1 is proved.

References

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