FUZZY CONVEX FUNCTIONS: A RUDIMENTARY THEORY AND ITS USE IN FUZZY-VALUED INEQUALITIES

ZHOU Feiyue
Department of Mathematics, Xiangtan Teachers' College, Hunan, China

This paper intends to establish a rudimentary convexity theory for the fuzzy-valued functions of a fuzzy variable. Some operations on fuzzy convex functions are investigated and the convexity of fuzzy extension functions of ordinary convex functions is checked. Moreover, two fuzzy-valued inequalities have been proved by using this fuzzy convex function theory.

Keywords: Compact normal convex fuzzy set, Fuzzy convex region, Fuzzy number, Fuzzy convex function, Fuzzy-valued inequality.

1. Preliminaries

Throughout this paper, we shall denote by $\mathbf{R}$ the set of all real numbers, by $\mathbf{R}^+$ the set of all positive real numbers and by $\mathbf{R}^-$ the set of all negative real numbers. If $\tilde{a}$ is a fuzzy set on $\mathbf{R}^n$, $\tilde{a}(x)$ will denote the grade of membership of x in $\tilde{a}$, $S(\tilde{a})$ the support of $\tilde{a}$ and for each $r \in]0,1]$ $[\tilde{a}]_r$ the r-level set of $\tilde{a}$: $[\tilde{a}]_r = \{x \in \mathbf{R}^n : \tilde{a}(x) \geq r\}$.

If $\tilde{a}, \tilde{b}$ are two fuzzy sets on $\mathbf{R}^n$ and λ a real number, as special cases of Zadeh's extension principle [14] we define $\tilde{a}+\tilde{b}$ and $\lambda\tilde{a}$ as

$$(\tilde{a}+\tilde{b})(x) = \sup_{y+z=x} \tilde{a}(y) \wedge \tilde{b}(z) \quad \text{and} \quad (\lambda\tilde{a})(x) = \begin{cases} \tilde{a}(\frac{1}{\lambda}x), & \text{if } \lambda \neq 0, \\ \tilde{\theta}_a(x), & \text{if } \lambda = 0 \end{cases}$$

respectively, where θ is the origin of $\mathbf{R}^n$, $a = \sup\{\tilde{a}(x): x \in \mathbf{R}^n\}$ and $\tilde{\theta}_a$ is the fuzzy set defined by

$$\tilde{\theta}_a(x) = \begin{cases} a, & \text{if } x = \theta, \\ 0, & \text{if } x \neq \theta. \end{cases}$$

It is well known (see Nguyen's theory [9]) that for each $r \in]0,1]$ we have

$$[\tilde{a} + \tilde{b}]_r = [\tilde{a}]_r + [\tilde{b}]_r \quad \text{and} \quad [\lambda\tilde{a}]_r = \lambda[\tilde{a}]_r , \tag{1.1}$$

where $[\tilde{a}]_r + [\tilde{b}]_r = \{x+y: x \in [\tilde{a}]_r, y \in [\tilde{b}]_r\}$ and $\lambda[\tilde{a}]_r = \{\lambda x: x \in [\tilde{a}]_r\}$.

Let $\tilde{a}$ be a fuzzy set on $\mathbf{R}^n$. Following Zadeh [13] and Weiss [12] we shall call $\tilde{a}$ a compact normal convex fuzzy set if for each $r \in]0,1]$ $[\tilde{a}]_r$ is a non-empty compact convex subset of $\mathbf{R}^n$.

We shall denote by $\widetilde{\mathbf{R}^n}$ the family of all compact normal convex fuzzy sets on $\mathbf{R}^n$ and, for each non-empty ordinary convex subset $\mathbf{C} \subset \mathbf{R}^n$, by $\tilde{\mathbf{C}}$ the family of all compact normal convex fuzzy sets with supports contained in $\mathbf{C}$: $\tilde{\mathbf{C}} = \{\tilde{a} \in \widetilde{\mathbf{R}^n}: S(\tilde{a}) \subset \mathbf{C}\}$.

Definition 1.1. Let $\mathbf{C}$ be a non-empty ordinary convex subset of $\mathbf{R}^n$. A non-empty subset $\underset{\sim}{D}$ of $\tilde{\mathbf{C}}$ is called a fuzzy convex region on $\mathbf{C}$ if and only if for all $\tilde{a}, \tilde{b} \in \underset{\sim}{D}$ and $0<\lambda<1$ we have $\lambda\tilde{a} + (1-\lambda)\tilde{b} \in \underset{\sim}{D}$.

Note that every singleton of $\mathbf{C}$ can also be regarded as an element of $\tilde{\mathbf{C}}$, so that each non-empty ordinary convex subset of $\mathbf{C}$ is a fuzzy convex region on $\mathbf{C}$. Further note that, for a fuzzy set $\tilde{a} \in \widetilde{\mathbf{R}^n}$, $\tilde{a} \in \tilde{\mathbf{C}}$ if and only if $[\tilde{a}]_r \subset \mathbf{C}$ for all $r \in]0,1]$, thus it follows from (1.1) that for any $\tilde{a}, \tilde{b} \in \tilde{\mathbf{C}}$ and $0<\lambda<1$ we have $\lambda\tilde{a} + (1-\lambda)\tilde{b} \in \tilde{\mathbf{C}}$, hence $\tilde{\mathbf{C}}$ itself is a fuzzy convex region on $\mathbf{C}$.

Now let us recall some knowledge about fuzzy numbers. We shall only consider the class of all compact normal convex fuzzy numbers. Namely, a fuzzy set $\tilde{a}$ on $\mathbf{R}$ is called a fuzzy number if and only if for every $r \in]0,1]$ $[\tilde{a}]_r$ is a non-empty bounded closed interval. Thus, in our case, $\widetilde{\mathbf{R}}$ is the set of all fuzzy numbers, $\widetilde{\mathbf{R}}^+$ (resp. $\widetilde{\mathbf{R}}^-$) is the set of all positive (resp. negative) fuzzy numbers.

For the arithmetic operations on $\widetilde{\mathbf{R}}$ we shall follow those of [1,2,7 and 8], namely, for $\tilde{a},\tilde{b} \in \widetilde{\mathbf{R}}$ we have

$$(\tilde{a}+\tilde{b})(x) = \sup\{\tilde{a}(y)\wedge\tilde{b}(z): y \in S(\tilde{a}), z \in S(\tilde{b}), y+z=x\},$$

$$(\tilde{a}-\tilde{b})(x) = \sup\{\tilde{a}(y)\wedge\tilde{b}(z): y \in S(\tilde{a}), z \in S(\tilde{b}), y-z=x\},$$

$$(\tilde{a}\cdot\tilde{b})(x) = \sup\{\tilde{a}(y)\wedge\tilde{b}(z): y \in S(\tilde{a}), z \in S(\tilde{b}), y\cdot z=x\},$$

and if $0 \notin S(\tilde{b})$,

$$(\tilde{a}/\tilde{b})(x) = \sup\{\tilde{a}(y)\wedge\tilde{b}(z): y \in S(\tilde{a}), z \in S(\tilde{b}), y/z=x\},$$

where we admit $\sup\phi=0$ for the empty set ϕ.

It is known that both addition and multiplication are associative and commutative. Furthermore, one can easily check that for any $\tilde{a},\tilde{b} \in \widetilde{\mathbf{R}}$ and $\lambda \in \mathbf{R}$ we have

$$\lambda(\tilde{a}+\tilde{b}) = \lambda\tilde{a}+\lambda\tilde{b}. \tag{1.2}$$

and for any $\tilde{a} \in \widetilde{\mathbf{R}}$, if both $\alpha,\beta \in \mathbf{R}^+$ or both $\alpha,\beta \in \mathbf{R}^-$, we have

$$(\alpha+\beta)\tilde{a} = \alpha\tilde{a}+\beta\tilde{a}. \tag{1.3}$$

We shall also use the following proposition which was given by Dubois and Prade [1,2]:

Proposition 1.1. Provided that $\tilde{a}$ is either a positive or a negative fuzzy number and that $\tilde{b}$ and $\tilde{c}$ are together positive or negative fuzzy numbers, then

$$\tilde{a}\cdot(\tilde{b}+\tilde{c}) = \tilde{a}\cdot\tilde{b}+\tilde{a}\cdot\tilde{c}.$$

In order to define fuzzy convex functions, we have to use an inequality relation between fuzzy numbers. The following definition is very popular (see, for examples, [3 − 7] and [10]):

Definition 1.2. Let $\tilde{a},\tilde{b} \in \widetilde{\mathbf{R}}$. Then $\tilde{a} \preccurlyeq \tilde{b}$, or write $\tilde{b} \succcurlyeq \tilde{a}$, if and only if for all $r \in]0,1]$ we have $\inf[\tilde{a}]_r \leqslant \inf[\tilde{b}]_r$ and $\sup[\tilde{a}]_r \leqslant \sup[\tilde{b}]_r$.

For the properties of $\leqslant$, we give the following proposition which will be used in the sequel.

Proposition 1.2. Let $\tilde{a},\tilde{b},\tilde{c},\tilde{d} \in \tilde{\mathbf{R}}$ and $\lambda \in \mathbf{R}$.

(1). If $\tilde{a} \leqslant \tilde{b}$ and $\tilde{b} \leqslant \tilde{c}$, then $\tilde{a} \leqslant \tilde{c}$.

(2). If $\tilde{a} \leqslant \tilde{b}$ and $\tilde{c} \leqslant \tilde{d}$, then $\tilde{a}+\tilde{c} \leqslant \tilde{b}+\tilde{d}$.

(3). If $\tilde{a} \leqslant \tilde{b}$, then for $\lambda \geqslant 0$ we have $\lambda\tilde{a} \leqslant \lambda\tilde{b}$ and for $\lambda \leqslant 0$ $\lambda\tilde{a} \geqslant \lambda\tilde{b}$.

(4). If $\tilde{a} \leqslant \tilde{b}$ and $\tilde{a},\tilde{b} \in \widetilde{\mathbf{R}^{+}} \cup \widetilde{\mathbf{R}^{-}}$, then for $\tilde{c} \in \widetilde{\mathbf{R}^{+}}$ we have $\tilde{c}\cdot\tilde{a} \leqslant \tilde{c}\cdot\tilde{b}$ and for $\tilde{c} \in \widetilde{\mathbf{R}^{-}}$ $\tilde{c}\cdot\tilde{a} \geqslant \tilde{c}\cdot\tilde{b}$.

Finally we conclude this preliminary section with a definition:

Definition 1.3. Let $\underset{\sim}{E}$ be a subset of $\tilde{\mathbf{R}}$. A mapping f from $\underset{\sim}{E}$ to $\tilde{\mathbf{R}}$ is said to be non-decreasing on $\underset{\sim}{E}$ if and only if for all $\tilde{a},\tilde{b} \in \underset{\sim}{E}$ such that $\tilde{a} \leqslant \tilde{b}$ we have $f(\tilde{a}) \leqslant f(\tilde{b})$.

2. Fuzzy convex functions

Throughout this section, $\underset{\sim}{D}$ will always denote a fuzzy convex region on $\mathbf{R}^n$.

Definition 2.1. A mapping f from $\underset{\sim}{D}$ to $\tilde{\mathbf{R}}$ is called a fuzzy convex function, or say that f is F-convex on $\underset{\sim}{D}$, if and only if for all $\tilde{a},\tilde{b} \in \underset{\sim}{D}$ and $0<\lambda<1$ we have

$$f(\lambda\tilde{a}+(1-\lambda)\tilde{b}) \leqslant \lambda f(\tilde{a})+(1-\lambda)f(\tilde{b}). \tag{2.1}$$

Definition 2.2. A mapping f from $\underset{\sim}{D}$ to $\tilde{\mathbf{R}}$ is called a fuzzy concave function, or say that f is F-concave on $\underset{\sim}{D}$, if and only if for all $\tilde{a},\tilde{b} \in \underset{\sim}{D}$ and $0<\lambda<1$ the inverse inequality of (2.1) holds.

It is easy to see from (1.2) and Proposition 1.2 (3) that a mapping f from $\underset{\sim}{D}$ to $\tilde{\mathbf{R}}$ is F-concave iff $-f$ is F-convex. Thus in the following we shall mainly discuss fuzzy convex functions.

Let $\mathbf{C}$ be a non-empty ordinary convex subset of $\mathbf{R}^n$ and f be an ordinary convex function on $\mathbf{C}$. Since $\mathbf{C}$ is a special fuzzy convex region and $\mathbf{R}$ is a set of special fuzzy numbers, of course $f:\mathbf{C}\to\mathbf{R}$ is a special fuzzy convex function. Furthermore, Zadeh's extension principle [14] tells us that, for each fuzzy convex region $\underset{\sim}{D}$ on $\mathbf{C}$, f can be extended as a mapping from $\underset{\sim}{D}$ to the family of all fuzzy sets on $\mathbf{R}$ such that for $\tilde{a}\in\underset{\sim}{D}$

$$f(\tilde{a})(t) = \sup_{x\in f^{-1}(t)} \tilde{a}(x) \quad \text{for } t\in\mathbf{R}, \tag{2.2}$$

and Nguyen [9] tells us that in the case f is continuous on $\mathbf{C}$ we have

$$[f(\tilde{a})]_r = f([\tilde{a}]_r) \quad \text{for } r\in]0,1],$$

and thus, by the compactness and convexity of $[\tilde{a}]_r$, for each $r\in]0,1]$ $[f(\tilde{a})]_r$ is a bounded closed interval, namely the fuzzy extension function defined by (2.2) is in fact a mapping from $\underset{\sim}{D}$ to $\tilde{\mathbf{R}}$.

Theorem 2.1. Let $\mathbf{C}$ be a non-empty ordinary convex subset of $\mathbf{R}^n$ and f be an ordinary convex function on $\mathbf{C}$. If f is continuous on $\mathbf{C}$, then for any fuzzy convex region $\underset{\sim}{D}\subset\tilde{\mathbf{C}}$ the fuzzy extension function $f:\underset{\sim}{D}\to\tilde{\mathbf{R}}$ defined by (2.2) is a fuzzy convex function.

Proof. Let $\tilde{a},\tilde{b}\in\underset{\sim}{D}$ and $0<\lambda<1$. Since for each $r\in]0,1]$ $[\tilde{a}]_r,[\tilde{b}]_r$ and $[\lambda\tilde{a}+(1-\lambda)\tilde{b}]_r = \lambda[\tilde{a}]_r+(1-\lambda)[\tilde{b}]_r$ are all non-empty compact subset of $\mathbf{C}$, and since f is continuous, there should exist $x_{min}\in[\tilde{a}]_r, y_{min}\in[\tilde{b}]_r$ and $\lambda x^*+(1-\lambda)y^*\in[\lambda\tilde{a}+(1-\lambda)\tilde{b}]_r$ ($x^*\in[\tilde{a}]_r$, $y^*\in[\tilde{b}]_r$) such that

$$f(x_{min}) = \inf f([\tilde{a}]_r), \quad f(y_{min}) = \inf f([\tilde{b}]_r) \tag{2.3}$$

and

$$f(\lambda x^*+(1-\lambda)y^*) = \sup f([\lambda\tilde{a}+(1-\lambda)\tilde{b}]_r) . \tag{2.4}$$

Note that we always have

$$f(\lambda x_{min}+(1-\lambda)y_{min}) \geq \inf f([\lambda\tilde{a}+(1-\lambda)\tilde{b}]_r), \tag{2.5}$$

$$f(x^*) \leq \sup f([\tilde{a}]_r) \quad \text{and} \quad f(y^*) \leq \sup f([\tilde{b}]_r), \tag{2.6}$$

and note that also by applying Nguyen's theory [9] we have

$$\inf f([\tilde{a}]_r) = \inf [f(\tilde{a})]_r, \quad \inf f([\tilde{b}]_r) = \inf [f(\tilde{b})]_r, \tag{2.7}$$

$$\sup f([\tilde{a}]_r) = \sup [f(\tilde{b})]_r, \quad \sup f([\tilde{b}]_r) = \sup [f(\tilde{b})]_r, \tag{2.8}$$

$$\sup f([\lambda\tilde{a} + (1-\lambda)\tilde{b}]_r) = \sup [f(\lambda\tilde{a} + (1-\lambda)\tilde{b})]_r, \tag{2.9}$$

and

$$\inf f([\lambda\tilde{a} + (1-\lambda)\tilde{b}]_r) = \inf [f(\lambda\tilde{a} + (1-\lambda)\tilde{b})]_r. \tag{2.10}$$

Moreover, by the convexity of f we have

$$f(\lambda x_{min} + (1-\lambda)y_{min}) \leq \lambda f(x_{min}) + (1-\lambda) f(y_{min}), \tag{2.11}$$

and

$$f(\lambda x^* + (1-\lambda)y^*) \leq \lambda f(x^*) + (1-\lambda) f(y^*). \tag{2.12}$$

Thus we have by (2.10),(2.5),(2.11),(2.3) and (2.7)

$$\inf [f(\lambda\tilde{a}+(1-\lambda)\tilde{b})]_r \leq \lambda(\inf [f(\tilde{a})]_r) + (1-\lambda)(\inf [f(\tilde{b})]_r), \tag{2.13}$$

and by (2.9),(2.4),(2.12),(2.6) and (2.8)

$$\sup [f(\lambda\tilde{a}+(1-\lambda)\tilde{b})]_r \leq \lambda(\sup [f(\tilde{a})]_r) + (1-\lambda)(\sup [f(\tilde{b})]_r). \tag{2.14}$$

Furthermore, one can easily check that we have

$$\lambda(\inf [f(\tilde{a})]_r) + (1-\lambda)(\inf [f(\tilde{b})]_r) = \inf(\lambda [f(\tilde{a})]_r + (1-\lambda)[f(\tilde{b})]_r) = \inf [\lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b})]_r, \tag{2.15}$$

and

$$\lambda(\sup [f(\tilde{a})]_r) + (1-\lambda)(\sup [f(\tilde{b})]_r) = \sup(\lambda [f(\tilde{a})]_r + (1-\lambda)[f(\tilde{b})]_r) = \sup [\lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b})]_r. \tag{2.16}$$

Consequently we have by (2.13),(2.14),(2.15) and (2.16)

$$\inf [f(\lambda\tilde{a} + (1-\lambda)\tilde{b})]_r \leq \inf [\lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b})]_r$$

and

$$\sup [f(\lambda\tilde{a} + (1-\lambda)\tilde{b})]_r \leq \sup [\lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b})]_r,$$

namely we have

$$f(\lambda\tilde{a} + (1-\lambda)\tilde{b}) \lesssim \lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b}).$$

This completes the proof.

Now let us discuss some fundamental operations on fuzzy convex functions.

Theorem 2.2. If both f and g are F-convex on $\underset{\sim}{D}$, then so are $f+g$ and λf ($\lambda \geqslant 0$). Namely the family of all fuzzy convex functions on $\underset{\sim}{D}$ is closed under positive linear combinations.

Proof. Immediate from (1.2) and Proposition 1.2 (2) and (3).

Theorem 2.3. If f is F-convex on $\underset{\sim}{D}$ and if for all $\tilde{a} \in \underset{\sim}{D}$ we have $f(\tilde{a}) \in \widetilde{R}^{+}$ or for all $\tilde{a} \in \underset{\sim}{D}$ we have $f(\tilde{a}) \in \widetilde{R}^{-}$, then for any $\tilde{\lambda} \in \widetilde{R}^{+}$, $\tilde{\lambda} \cdot f$ is a fuzzy convex function on $\underset{\sim}{D}$.

Proof. Immediate from Proposition 1.1 and Proposition 1.2 (4).

Theorem 2.4. If f is a fuzzy convex function from $\underset{\sim}{D}$ to a fuzzy convex region $\underset{\sim}{E} \subset \widetilde{\mathbf{R}}$, then for any a non-decreasing fuzzy convex function g on $\underset{\sim}{E}$, the compound function $g(f(\tilde{x}))$ is F-convex on $\underset{\sim}{D}$.

Proof. Let $\tilde{a}, \tilde{b} \in \underset{\sim}{D}$ and $0<\lambda<1$. Since it is known that we have

$$f(\lambda \tilde{a} + (1-\lambda)\tilde{b}) \leqslant \lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b}),$$

and since g is non-decreasing, of course we have

$$g(f(\lambda \tilde{a} + (1-\lambda)\tilde{b})) \leqslant g(\lambda f(\tilde{a}) + (1-\lambda) f(\tilde{b})),$$

and hence by the F-convexity of g we can get

$$g(f(\lambda \tilde{a} + (1-\lambda)\tilde{b}) \leqslant \lambda g(f(\tilde{a})) + (1-\lambda) g(f(\tilde{b})).$$

This completes the proof.

In order to obtain some more operations on fuzzy convex functions we need the following lemma:

Lemma 2.5. Let $\mathbf{I}$ be a non-empty interval of $\mathbf{R}$ and f be an ordinary non-decreasing function on $\mathbf{I}$. If f is continuous on $\mathbf{I}$, then for any subset $\underset{\sim}{E}$ of $\widetilde{\mathbf{I}}$, the fuzzy extension function $f:\underset{\sim}{E}\to\widetilde{\mathbf{R}}$ defined by (2.2) is also non-decreasing.

Proof. Straightforward.

Theorem 2.6. Let $\mathbf{I}$ be a non-empty interval of $\mathbf{R}$, and let f be a fuzzy convex function from $\underset{\sim}{D}$ to $\widetilde{\mathbf{I}}$. Then for any a non-decreasing ordinary convex function g on $\mathbf{I}$, the compound function $g(f(\widetilde{x}))$ is also a fuzzy convex function on $\underset{\sim}{D}$.

Proof. Immediate from Lemma 2.5, Theorem 2.1 and Theorem 2.4.

Corollary 2.7. If f is F-convex on $\underset{\sim}{D}$, then so is $e^{f(\widetilde{x})}$.

Corollary 2.8. If f is F-convex and positive on $\underset{\sim}{D}$, then so is $[f(\widetilde{x})]^p$ for $p>1$.

Corollary 2.9. If f is F-concave and positive on $\underset{\sim}{D}$, then $1/f(x)$ is F-convex on $\underset{\sim}{D}$.

Proof. It is known that -f is F-convex on $\underset{\sim}{D}$ and for all $\widetilde{a}\in\underset{\sim}{D}$ we have $f(\widetilde{a})\in\widetilde{\mathbf{R}^-}$. Further we know that $g(t)=-1/t$ is an ordinary increasing convex function on $\mathbf{R}^-$. Hence by virtue of Theorem 2.6 the compound function $g(f(x))=-1/(-f(x))=1/f(x)$ is F-convex on $\underset{\sim}{D}$.

Finally we conclude our this fuzzy convex function theory with the following fuzzy extension of Jenson's inequality which will be useful in the theory of fuzzy-valued inequalities.

Theorem 2.10. Let f be a mapping from $\underset{\sim}{D}$ to $\widetilde{\mathbf{R}}$. Then f is F-convex on $\underset{\sim}{D}$ if and only if for any $\widetilde{a}_1,\cdots,\widetilde{a}_m\in\underset{\sim}{D}$

$$f(\lambda_1\widetilde{a}_1+\cdots+\lambda_m\widetilde{a}_m)\precsim\lambda_1 f(\widetilde{a}_1)+\cdots+\lambda_m f(\widetilde{a}_m)$$

whenever $\lambda_1\geqslant 0,\cdots,\lambda_m\geqslant 0,\ \lambda_1+\cdots+\lambda_m=1$.

Proof. Analogous to the proof of Jenson's inequality in the non-fuzzy case.

3. Applications to fuzzy-valued inequalities

In paper [15], we have established a rudimentary theory of fuzzy-valued inequalities and have extended four classical inequalities to fuzzy-valued case. In this section we shall now use our fuzzy convex function theory to prove two more fuzzy-valued inequalities.

Inequality 1. Let λ_{ij} ($i,j=1,\cdots,n$) be n^2 positive real numbers satisfying $\lambda_{i1}+\cdots+\lambda_{in}=1$ and $\lambda_{1j}+\cdots+\lambda_{nj}=1$ ($i,j=1,\cdots,n$) and let $\tilde{a}_1,\cdots,\tilde{a}_n \in \widetilde{\mathbf{R}}^+$. If $\tilde{b}_i=\lambda_{i1}\tilde{a}_1+\cdots+\lambda_{in}\tilde{a}_n$ ($i=1,\cdots,n$), then we have

$$\tilde{b}_1\cdot\cdots\cdot\tilde{b}_n \gtrsim \tilde{a}_1\cdot\cdots\cdot\tilde{a}_n \,. \tag{3.1}$$

Proof. Theorem 2.1 tells us that $f(\tilde{x})=-\ln\tilde{x}$ is F-convex on $\widetilde{\mathbf{R}}^+$. Hence for $i=1,\cdots,n$ we have by Theorem 2.10

$$\ln\tilde{b}_i=\ln(\lambda_{i1}\tilde{a}_1+\cdots+\lambda_{in}\tilde{a}_n) \gtrsim \lambda_{i1}\ln\tilde{a}_1+\cdots+\lambda_{in}\ln\tilde{a}_n .$$

So that by (1.3) we have

$$\ln\tilde{b}_1+\cdots+\ln\tilde{b}_n \gtrsim \ln\tilde{a}_1+\cdots+\ln\tilde{a}_n . \tag{3.2}$$

Since $\ln t$ is strictly increasing on $\mathbf{R}^+$, so one can easily check that for any $\tilde{a},\tilde{b} \in \widetilde{\mathbf{R}}^+$ we have

$$\ln\tilde{a}+\ln\tilde{b}=\ln(\tilde{a}\cdot\tilde{b}) \tag{3.3}$$

and that $\ln\tilde{a} \lesssim \ln\tilde{b}$ iff $\tilde{a} \lesssim \tilde{b}$. Therefore (3.2) is equivalent to (3.1).

Inequality 2. If $\tilde{a}_1,\cdots,\tilde{a}_n \in \widetilde{\mathbf{I}}$ ($\mathbf{I}=]0,1[$), then for any $\lambda_1,\cdots,\lambda_n \in \mathbf{R}^+$ such that $\lambda_1+\cdots+\lambda_n=1$ we have

$$\frac{\lambda_1}{1+\tilde{a}_1}+\cdots+\frac{\lambda_n}{1+\tilde{a}_n} \lesssim \frac{1}{1+\tilde{a}_1^{\lambda_1}\cdot\cdots\cdot\tilde{a}_n^{\lambda_n}} \tag{3.4}$$

Proof. Let $\tilde{b}_i=\ln\tilde{a}_i$. Then we have $\tilde{b}_i \in \widetilde{\mathbf{R}}^-$. Further it is easy to check that $\tilde{a}_i = e^{\tilde{b}_i}$ ($i = 1, \cdots, n$). Since $f(t) = -1/(1+e^t)$ is convex on $\mathbf{R}^-$, it follows at once from Theorem 2.1 and Theorem 2.10 that we have

$$\frac{\lambda_1}{1+e^{\tilde{b}_1}} + \cdots + \frac{\lambda_n}{1+e^{\tilde{b}_n}} \lesssim \frac{1}{1+e^{\lambda_1\tilde{b}_1+\cdots+\lambda_n\tilde{b}_n}} . \tag{3.5}$$

Note that for $\tilde{a}, \tilde{b} \in \tilde{\mathbf{R}}$ and $\lambda \in \mathbf{R}$ we also have $e^{\tilde{a}+\tilde{b}} = e^{\tilde{a}} \cdot e^{\tilde{b}}$ and $e^{\lambda\tilde{a}} = (e^{\tilde{a}})^{\lambda}$ (see [2]). Hence (3.5) is equivalent to (3.4).

References

[1] D.Dubois and H.Prade, Fuzzy real algebra: Some results, Fuzzy sets and Systems, 4(1979) 327 - 348.

[2] D.Dubois and H.Prade, Fuzzy Sets and Systems: Theory and Applications, (Academic Press, New York, 1980).

[3] D.Dubois and H.Prade, Comment on tolerance analysis using fuzzy sets "and" a procedure for multiple aspect decision making, Int. J. Systems Sci., 9(1978) 357 - 348.

[4] A.N.S.Freeling, Fuzzy sets and decision analysis, IEEE Trans. Systems, Man Cybernet. 10(1980) 341 - 354.

[5] R.Goetschel and W.Voxman, Eigen fuzzy number sets, Fuzzy Sets and Systems, 16(1985) 75 - 85.

[6] O.Kaleva and S.Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, 14(1984) 215 - 229.

[7] M.Mizumoto and K.Tanaka, Some properties of fuzzy numbers, in: M.M. Gupta, R.K.Ragade and R.R.Yager, Eds., Advances in Fuzzy Set Theory and Applications (North - Holland, Amsterdam, 1979) 153 - 164.

[8] M.Mizumoto and K.Tanaka, The four operations of arithmetic on fuzzy numbers, Systems - Comput. - Controls, 5(1976) 73 - 81.

[9] H.T.Nguyen, A note on the extension principle for fuzzy sets, J. Math. Anal. Appl. 64(1978) 369 - 380.

[10] J.Ramík and J.Římánek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, Fuzzy Sets and Systems, 16(1985) 123 - 138.

[11] R.T.Rockafellar, Convex Analysis (Princeton University Press, second printing, 1972).

[12] M.D.Weiss, Fixed points, separation and induced topologies for fuzzy sets, J. Math. Anal. Appl. 50(1975) 142 - 150.

[13] L.A.Zadeh, Fuzzy sets, Information and Control, 8(1965) 338 - 353.

[14] L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 1,2 and 3, Inform. Sci. 8, 199 - 249; 8, 301 - 357; 9, 43 - 80(1975).

[15] Zhou Feiyue, On fuzzy-valued inequalities, Fuzzy Sets and Systems, To appear.