t-NORM-BASED ADDITION OF FUZZY INTERVALS*

Róbert FULLÉR, Tibor KERESZTFALVI

Computer Center, Loránd Eötvös University
H-1502 Budapest 112, P.O.Box 157, Hungary

The aim of this paper is to provide new results regarding the effective practical computation of t-norm-based additions of LR-fuzzy intervals.

Keywords: t-norm; extension principle; fuzzy interval

1. Introduction

The present paper is devoted to the derivation of exact calculation formulas for a t-norm-based addition of LR-fuzzy intervals (the addition rule of LR-fuzzy intervals is well-known in the case of "min"-norm).

A very important feature of the approach by t-norms is that it provides means of controlling the growth of uncertainty in calculations, and prevents variables from simultaneous shift off their most significant values. In this respect, the various addition formulas yield practical tools for achieving this control, and are very meaningful.

Our results are connected with those presented in [2] and we generalize and extend them. Namely, we shall determine the exact membership function of the t-norm-based sum of fuzzy intervals, in the case of Archimedean t-norm having strictly convex additive generator function and fuzzy intervals with concave shape functions.

It should be noted that the class of t-norms mentioned above

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is wide enough (it contains among others: Yager's, Dombi's, Hamacher's, Schweizer's, Frank's, Weber's parametrized t-norms with adequate parameters) to be useful in practical computations.

2. Definitions

A fuzzy interval $\tilde{a}$ is a fuzzy set of real numbers $\mathbb{R}$ with a continuous, compactly supported, unimodal and normalized membership function. It is known [1], that any fuzzy interval $\tilde{a}$ can be described with the following membership function:

$$\tilde{a}(x) = \begin{cases} 1 & \text{if } a \le x \le b \\ L\left(\frac{a-x}{\alpha}\right) & \text{if } a-\alpha \le x \le a,\ \alpha > 0 \\ R\left(\frac{x-a}{\beta}\right) & \text{if } b \le x \le b+\beta,\ \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $[a,b]$ is the peak of $\tilde{a}$; a and b are the lower and upper modal values; L and R are shape functions: $[0,1] \to [0,1]$, with $L(0)=R(0)=1$ and $L(1)=R(1)=0$ which are non-increasing, continuous mappings. We shall call these fuzzy intervals of LR-type and use the notation $\tilde{a}=(a,b,\alpha,\beta)_{LR}$. The support of $\tilde{a}$ is exactly $[a-\alpha, b+\beta]$.

Recall that a triangular norm (t-norm for short) T is Archimedean iff T is continuous and $T(x,x)<x$ for all $x \in (0,1)$.

Every Archimedean t-norm T is representable by a continuous and decreasing function $f:[0,1] \to [0,\infty]$ with $f(1)=0$ and

$$T(x,y) = f^{[-1]}(f(x)+f(y))$$

where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) := \begin{cases} f^{-1}(y) & \text{if } y \in [0,f(0)] \\ 0 & \text{if } y \in [f(0),\infty] \end{cases}.$$

The function f is the additive generator of T.

If T is a t-norm and $\tilde{a}_1$, $\tilde{a}_2$ are fuzzy sets of the real line (i.e. fuzzy quantities) then their T-sum $\tilde{A}_2:=\tilde{a}_1+\tilde{a}_2$ is defined by

$$\tilde{A}_2(z) = \sup_{x_1+x_2=z} T\left[\tilde{a}_1(x_1),\tilde{a}_2(x_2)\right] , \quad z\in\mathbb{R},$$

which expression can be written in the form

$$\tilde{A}_2(z) = \sup_{x_1+x_2=z} f^{[-1]}\left[f(\tilde{a}_1(x_1))+f(\tilde{a}_2(x_2))\right] ,$$

supposing that f is the additive generator of T.

By the associativity of T, the membership function of the T-sum $\tilde{A}_n:=\tilde{a}_1+\ldots+\tilde{a}_n$ of fuzzy quantities $\tilde{a}_1,\ldots,\tilde{a}_n$ can be written as

$$\tilde{A}_n(z) = \sup_{x_1+\ldots+x_n=z} f^{[-1]}\left[f(\tilde{a}_1(x_1))+\ldots+f(\tilde{a}_n(x_n))\right] .$$

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$\tilde{A}_n(z) = f^{[-1]}\left(\inf_{x_1+\ldots+x_n=z}\left[f(\tilde{a}_1(x_1))+\ldots+f(\tilde{a}_n(x_n))\right]\right) .$$

3. t-norm-based addition of fuzzy intervals

In the following theorem we shall determine a class of t-norms in which the addition of fuzzy intervals is very simple.

Theorem 1. Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i=(a_i,b_i,\alpha,\beta)_{LR}$, $i=1,\ldots,n$ be fuzzy intervals of LR-type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function then the membership function of the T-sum

$\tilde{A}_n=\tilde{a}_1+...+\tilde{a}_n$ is

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if} \quad A_n \le z \le B_n \\ f^{[-1]}\left[n \cdot f\left[L\left(\frac{A_n-z}{n \cdot \alpha}\right)\right]\right] & \text{if} \quad A_n - n\cdot\alpha \le z \le A_n \\ f^{[-1]}\left[n \cdot f\left[R\left(\frac{z-B_n}{n \cdot \beta}\right)\right]\right] & \text{if} \quad B_n \le z \le B_n + n\cdot\beta \\ 0 & \text{otherwise} \end{cases}$$

where $A_n=a_1+...+a_n$ and $B_n=b_1+...+b_n$.

Remark. It should be noted, that from the concavity of shape functions it follows that the fuzzy intervals in question can not have infinite support.

3. Applications

We shall illustrate Theorem 1 for Yager's, Dombi's and Hamacher's parametrized t-norm.

For simplicity we shall restrict our consideration to the case of symmetric fuzzy numbers $\tilde{a}_i=(a_i,a_i,\alpha,\alpha)_{LL}$ $i=1,...,n$.

Denoting $\sigma_n := \frac{|A_n-z|}{n\cdot\alpha}$, we get the following formulas for the membership function of t-norm-based sum $\tilde{A}_n=\tilde{a}_1+...+\tilde{a}_n$:

(i) Yager's t-norm with $p>1$:

$$T(x,y) = 1-\min\left\{1, \sqrt[p]{(1-x)^p+(1-y)^p}\right\}$$

which has additive generator

$$f(x) = (1-x)^p$$

$$\tilde{A}_n(z) = \begin{cases} 1-n^{1/p}\cdot(1-L(\sigma_n)) & \text{if} \quad \sigma_n < L^{-1}(1-n^{-1/p}) \\ 0 & \text{otherwise} \end{cases}$$

(ii) Hamacher's t-norm with $p \leq 2$:

$$T(x,y) = \frac{x \cdot y}{p+(1-p)\cdot(x+y-x\cdot y)}$$

having additive generator

$$f(x) = \ln\frac{p+(1-p)\cdot x}{x}$$

$$\tilde{A}_n(z) = \begin{cases} \dfrac{p}{\left[\dfrac{p+(1-p)\cdot L(\sigma_n)}{L(\sigma_n)}\right]^n - 1 + p} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise} \end{cases}$$

(iii) Dombi's t-norm with $p > 1$:

$$T(x,y) = \frac{1}{1+\sqrt[p]{(1/x-1)^p+(1/y-1)^p}}$$

with additive generator

$$f(x) = \left[\frac{1-x}{x}\right]^p$$

$$\tilde{A}_n(z) = \begin{cases} \dfrac{1}{1+n^{1/p}\cdot\left[\dfrac{1}{L(\sigma_n)}-1\right]} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig.1.);}$$

(iv) Algebraic t-norm (i.e. the Hamacher's t-norm with p=1):

$T(x,y) = x \cdot y$ having additive generator $f(x) = -\ln x$

$$\tilde{A}_n(z) = L^n(\sigma_n) \qquad z \in \mathbb{R} .$$

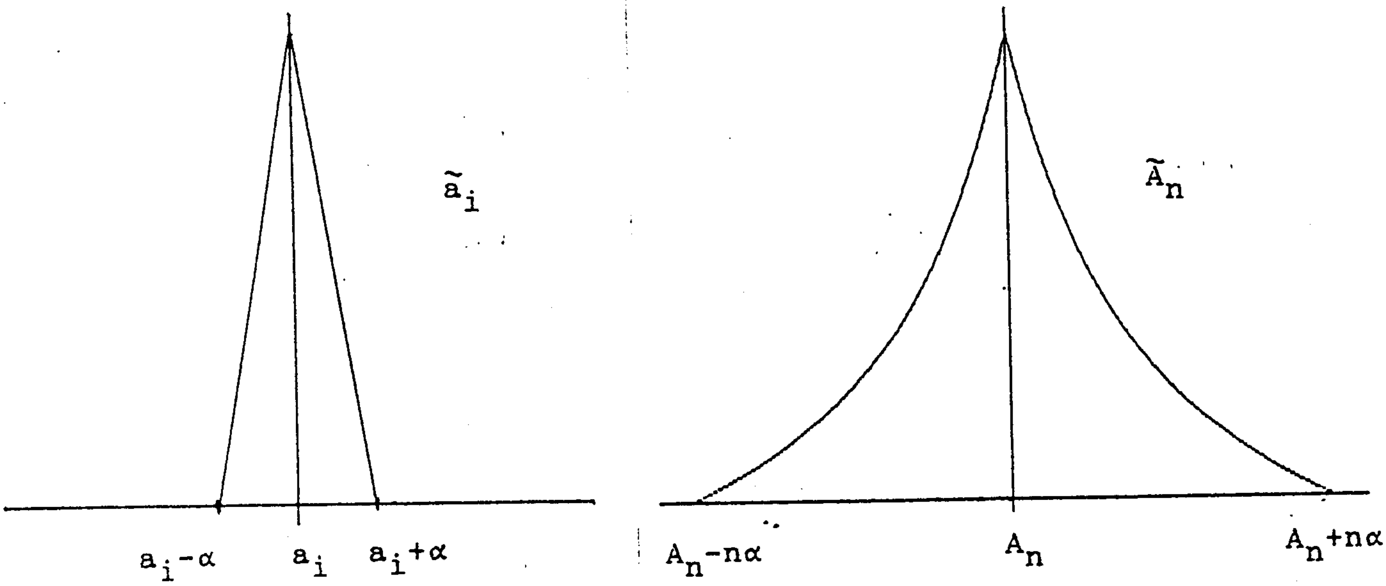

Figure 1. Addition of symmetric triangular fuzzy numbers via Dombi's t-norm (p=3/2).

References

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[3] D.Dubois and H.Prade, Inverse Operations for Fuzzy Numbers, Proc. of IFAC Symp. on Fuzzy Information, Knowledge, Representation and Decision Analysis, Marseille, 1983 399-404.