A NOTE ON CONVEX FUZZY SETS

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This note is to give two weak conditions that the fuzzy closed set is convex fuzzy set.

Key words: Convex Fuzzy set, Fuzzy closed set.

1. Introduction

In the basic and classical paper[1], where the important concept of fuzzy set was first introduced, Zadeh developed a basic framwork to treat mathematically the fuzzy pheaomena or systems which, due to intrinsic indefiniteness, cannot themselves be characterized precisely. He pays special attention to the investigation of the convex fuzzy sets which covers hearly the second half of the space of the paper. The aim of this note is to give two weak conditions that the fuzzy closed set is convex fuzzy set.

For simplicity, we consider only the fuzzy sets defined on the Eucliden space in this note. But it is not difficult to generalize the results obtained in the note to the case that fuzzy sets are defined in linear space over real field or comlex field.

2. Prelimiaries

Throughout this paper E will denote the n-dimensional Euclidean space R^n, I denotes the interval [0,1], and $\dot{I}$ denotes (0,1). Fuzzy sets and values in I will be denoted by lower case Greek letters and we shall make no difference between notations for a fuzzy set with a constant value and that value itself.

Definition 1. The fuzzy set λ on E is said to be convex fuzzy set iff for all $x,y\in E$ and $a\in I$,

$$\lambda(ax+(1-a)y)\geqslant\lambda(x)\wedge\lambda(y)$$

It is easy to see that λ is a convex fuzzy set iff for all $a\in I$, $\lambda^{-1}[a,1]$ is convex.

Definiton 2. The fuzzy set λ on E is said to be fuzzy closed set iff for all $a\in I$, $\lambda^{-1}[a,1]$ is closed.

3. Main results

This section is to give the main results of this note. That is, we obtain two weak conditions that the fuzzy closed set is convex fuzzy set.

Theorem 1. Let λ be fuzzy closed set on E, then λ is convex fuzzy set iff for all $x,y \in E$, there exists an $a \in \dot{I}$ (a depends on x,y), such that

$$\lambda(ax+(1-a)y) \geqslant \lambda(x) \wedge \lambda(y)$$

Proof. It is sufficient to show that $\lambda^{-1}[a,1]$ is convex set for all $a \in I$. In fact, if that is not true, then there exists a $b \in I$, $\lambda^{-1}[b,1]$ is not convex set, thus there exist $x^1, x^2 \in \lambda^{-1}[b,1], a_0 \in I$ such that

$$x^{a_0} = a_0 x^1 + (1-a_0)x^2 \notin \lambda^{-1}[b,1]$$

Let

$$A = \lambda^{-1}[b,1] \cap [x^1, x^{a_0}], \qquad B = \lambda^{-1}[b,1] \cap [x^{a_0}, x^2]$$

where

$$[x^1, x^{a_0}] = \{ax^1 + (1-a)x^{a_0} \mid a \in I\}, [x^{a_0}, x^2] = \{ax^{a_0} + (1-a)x^2 \mid a \in I\}$$

Since λ be fuzzy closed set on E, it is easy to see that A and B are bound closed sets, and $x^{a_0} \notin A$, $x^{a_0} \notin B$. Thus there exists $\bar{x} \in A$ and $\bar{\bar{x}} \in B$ such that

$$\min_{x \in A} \| x - x^{a_0} \| = \| \bar{x} - x^{a_0} \|, \qquad \min_{x \in B} \| x - x^{a_0} \| = \| \bar{\bar{x}} - x^{a_0} \|$$

Hence, we have

$$\lambda^{-1}[b,1] \cap (\bar{x}, x^{a_0}] = \phi \quad \text{and} \quad \lambda^{-1}[b,1] \cap [x^{a_0}, \bar{\bar{x}}) = \phi$$

that is

$$\lambda^{-1}[b,1] \cap (\bar{x}, \bar{\bar{x}}) = \phi .$$

On the other hand, by the hypothesis of the theorem, there exists an $a \in \dot{I}$, such that

$$\lambda(a\bar{x} + (1-a)\bar{\bar{x}}) \geqslant \lambda(\bar{x}) \wedge \lambda(\bar{\bar{x}})$$

Since

$$\bar{x} \in \lambda^{-1}[b,1], \qquad \bar{\bar{x}} \in \lambda^{-1}[b,1]$$

we have

$$\lambda(a\bar{x} + (1-a)\bar{\bar{x}}) \geqslant b$$

that is

$$a\bar{x} + (1-a)\bar{\bar{x}} \in \lambda^{-1}[b,1]$$

This contradicts that $\lambda^{-1}[b,1]\cap(\bar{x},\bar{\bar{x}})=\phi$.
Thus, we prove $\lambda^{-1}[a,1]$ is convex set for all $a\in I$.

Corollary 1. Suppose that λ be fuzzy closed set on E, then λ is convex fuzzy set iff there exists an $a\in\dot{I}$, $\lambda(ax+(1-a)y)\geqslant\lambda(x)\wedge\lambda(y)$ for all $x,y\in E$.

Corollary 2. Let λ be fuzzy closed set on E, then λ is convex fuzzy set iff for all $x,y\in E$,

$$\lambda(\tfrac{1}{2}x+\tfrac{1}{2}y)\geqslant\lambda(x)\wedge\lambda(y).$$

Theorem 2. Let λ be closed fuzzy set on E, and there exists an $a\in\dot{I}$, such that

$$\lambda(ax+(1-a)y)>\lambda(x)\quad\text{for all}\quad x,y\in E$$

satisfying

$$\lambda(x)<\lambda(y).$$

Then λ be a convex fuzzy set.

Proof. By theorem 1, it is sufficien to show that for all $x,y\in E$, there exists a $b\in\dot{I}$, such that

$$\lambda(bx+(1-b)y)\geqslant\lambda(x)\wedge\lambda(y).$$

Suppose, on the contrary, that there exists $x^1, x^2\in E$, such that for all $b\in\dot{I}$,

$$\lambda(bx^1+(1-b)x^2)<\lambda(x^1)\wedge\lambda(x^2)\qquad(1)$$

If $\lambda(x^1)\neq\lambda(x^2)$, without loss of generality we can suppose that $\lambda(x^1)<\lambda(x^2)$. By the hypothesis of the theorem, we have

$$\lambda(ax^1+(1-a)x^2)>\lambda(x^1).$$

This leads to a contridiction.

If $\lambda(x^1)=\lambda(x^2)$. By inequality (1) and the hypothesis of the theorem, we obtain

$$\begin{aligned}\lambda(x^2)&>\lambda((ab+1-a)x^1+a(1-b)x^2)\\&=\lambda(a(bx^1+(1-b)x^2)+(1-a)x^1)\\&>\lambda(bx^1+(1-b)x^2)\qquad\forall\, b\in\dot{I}\qquad(2)\end{aligned}$$

By theorem conditions again, we have

$$\lambda\left(a\{(ab+(1-a))x^1+a(1-b)x^2\}+(1-a)x^2\right)$$

$$>\lambda((ab+1-a)x^1+a(1-b)x^2) \qquad \text{(By (2))}$$

$$>\lambda(bx^1+(1-b)x^2) \qquad \forall\ b\in \dot{I} \qquad \text{(By (1))}$$

Let $b \triangleq \bar{b}=a/(1+a)\in \dot{I}$, above inequality become

$$\lambda(\bar{b}x^1+(1-\bar{b})x^2)<\lambda(\bar{b}x^1+(1-\bar{b})x^2).$$

This leads to a contridiction. Thus inequality (1) is not true, by theorem 1, λ be convex fuzzy set on E.

corollary 3. Let λ is the fuzzy closed set, and for all $x,y\in E$, satisfying

$$\lambda(x)<\lambda(y)$$

we have

$$\lambda(\tfrac{1}{2}x+\tfrac{1}{2}y)>\lambda(x)$$

then λ be a convex fuzzy set.

References

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