The Relationship Between

Operations of Random Subsets and Fuzzy Subsets

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Abstract

On the base of the theoretical framework of [1], this paper approached the relationship between the operations of random subsets and fuzzy subsets——Cartetian, production induced operation, intersection, union and complement operations. It shows that there is a mapping from the collection of bivariate random vectors to the collection of projectable random intervals. We have also given a computing formula of the projected fuzzy set of the random interval which is the image of the bivariate random vector under our mapping.

1. Preparatory **Knowledge**

The definitions and notations of [1] relating to the paper are given as preparatory knowledge as follows:

U is a given basic space, $\mathcal{B}$ is a σ-algebra.

$$\forall u\in U, \quad C(u)\triangleq C_{\mathcal{B}}(u)\triangleq \{B\in\mathcal{B} \mid u\in B\} \tag{1.1}$$

$$\check{\mathcal{B}} \triangleq \sigma(\{C(u) \mid u\in U\}) \tag{1.2}$$

where $\sigma(\mathcal{A})$ is least σ-algebra containing $\mathcal{A}$.

$(\mathcal{B}, \check{\mathcal{B}})$ is a projectable measurable space.

Suppose that $(\Omega, \mathcal{A}, P)$ is a given probability space,

$$s: \Omega \to \mathcal{B} \tag{1.3}$$

is said to be projectable random subsets. On the U if s is $\mathcal{A}$-$\check{\mathcal{B}}$

measurable.

All projectable random subsets on the U is denoted by $S(\Omega, \mathcal{A}, P; U, \mathcal{B}, \check{\mathcal{B}})$ or S.

Let $s \in S$, $\underset{\smile}{s}$ is a fuzzy subset on the U. If $\forall u \in U$,

$$\mu_{\underset{\smile}{s}}(u) = P \circ s^{-1}(C(u)) = p(\{\omega \mid u \in S(\omega)\}), \tag{1.4}$$

then $\underset{\smile}{s}$ is said to be projectable fuzzy set of s.

$\mathcal{B}_0$ is the Borel field on R,

$$\delta \doteq \{[x_1, x_2] \mid \quad x_1, x_2 \in R, \text{ and } x_1 \le x_2\} \tag{1.5}$$

$s \in S(\Omega, \mathcal{A}, P; R, \mathcal{B}_0, \check{\mathcal{B}}_0)$ is called the random interval if $s \in \delta$.

2. Projectable measurable product space and product random sets

Definition 2.1 Suppose that $(\mathcal{B}_1, \check{\mathcal{B}}_1)$ is a projectable measurable space on U, $(\mathcal{B}_2, \check{\mathcal{B}}_2)$ is a projectable measurable space on V. $(\mathcal{B}_1 \times \mathcal{B}_2, (\mathcal{B}_1 \check{\times} \mathcal{B}_2))$ is said to be projectable measurable space on product space $U \times V$, where $\mathcal{B}_1 \times \mathcal{B}_2$ is product σ-algebra of the $\mathcal{B}_1$ and $\mathcal{B}_2$.

Let $s_1 \in S(\Omega, \mathcal{A}, P, U, \mathcal{B}_1, \check{\mathcal{B}}_1)$

$s_2 \in S(\Omega, \mathcal{A}, P; V, \mathcal{B}_2, \check{\mathcal{B}}_2)$.

The $s_1 \times s_2: \Omega \longrightarrow \mathcal{B}_1 \times \mathcal{B}_2$

$$\forall \omega \in \Omega, \ (s_1 \times s_2)(\omega) = s_1(\omega) \times s_2(\omega) \tag{2.1}$$

is said to be product random set of s_1 and s_2.

Theorem 2.1 $s_1 \times s_2 \in S(\Omega, \mathcal{A}, P, U \times V, \mathcal{B}_1 \times \mathcal{B}_2, (\mathcal{B}_1 \check{\times} \mathcal{B}_2))$, and

$$(s_1 \times s_2)^{-1}(C((u,v))) = s_1^{-1}(C(u)) \cap s_2^{-1}(C(u)) \tag{2.2}$$

$$\mu_{(s_1 \underset{\smile}{\times} s_2)}((u,v)) = P(s_1^{-1}(C(u)) \cap s_2^{-1}(C(v))) \tag{2.3}$$

Proof: we only prove (2.2).

Suppose $\omega \in (s_1 \times s_2)^{-1}(C((u,v)))$, then $s_1(\omega) \times s_2(\omega) \in C((u,v))$, so $(u,v) \in s_1(\omega) \times s_2(\omega)$, $u \in s_1(\omega)$ $v \in s_2(\omega)$

$s \in s_1^{-1}(C(u)) \cap s_2^{-1}(C(v))$.

Thus $(s_1 \times s_2)^{-1}(C((u,v))) = s_1^{-1}(C(u)) \cap s_2^{-1}(C(v))$.

On the contrary, suppose $\omega \in s_1^{-1}(C(u)) \cap s_2^{-1}(C(v))$. then $s_1(\omega) \in C(u)$ and $s_2(\omega) \in C(v)$.

So, $(u,v) \in s_1(\omega) \times s_2(\omega)$

$(u,v) \in (s_1 \times s_2)(\omega)$

$\omega \in (s_1 \times s_2)^{-1}(C((u,v)))$

Thus $s_1^{-1}(C(u)) \cap s_2^{-1}(C(v)) \subset (s_1 \times s_2)^{-1}(C((u,v)))$.

Therefore $s_1^{-1}(C(u)) \cap s_2^{-1}(C(V)) = (s_1 \times s_2)^{-1}(C((u,v)))$.

3. Induced projected measurable space and induced random sets

Suppose that $f: U \longrightarrow V$ satisfying

$\forall v \in V \quad f^{-1}(v)$ is a numberable subset on U at most.

Sometimes we regard also f as mapping from $\mathcal{P}(U)$ to $\mathcal{P}(V)$:

$$\forall A \in \mathcal{P}(U), \quad f(A) \triangleq \{ f(x) \mid x \in A \} \tag{3.1}$$

Definition 3.1 Suppose that $(\mathcal{B}, \check{\mathcal{B}})$ is a projectable meaurable space, $(\sigma(f(\mathcal{B})), \sigma(f(\check{\mathcal{B}})))$ is said to be projectable measurable space on V induced by f from $(\mathcal{B}, \check{\mathcal{B}})$, where $f(\mathcal{B}) = \{ f(B) \mid B \in \mathcal{B} \}$.

$$f(s): \Omega \longrightarrow \sigma(f(\mathcal{B}))$$

$$\forall \omega \in \Omega, \quad (f(s))(\omega) = f(s(\omega)) \tag{3.2}$$

is said to be projectable random sets on V induced by f from s.

Theorem 3.1 $f(s) \in S(\Omega, \mathcal{A}, P, V, \sigma(f(\mathcal{B})), \sigma(f(\check{\mathcal{B}})))$

and
$$(f(s))^{-1}(C(v)) = \begin{cases} \bigcup_{f(u)=v} s^{-1}(c(u)) & , \text{if } f^{-1}(v) \neq \phi \\ \phi & , \text{if } f^{-1}(v) = \phi \end{cases} \tag{3.3}$$

$$\forall v \in V \quad \mu_{\underset{\vee}{f(s)}}(v) = \begin{cases} P\left(\bigcup_{f(u)=v} s^{-1}(c(u))\right) & , \text{if } f^{-1}(v) \neq \phi \\ 0 & , \text{if } f^{-1}(v) = \phi \end{cases} \tag{3.4}$$

Proof: We only prove (3.3).

If $f^{-1}(v)=\phi$, we have $(f(s))^{-1}(C(v))=\phi$.

If $f^{-1}(v)\neq\phi$, $\omega\in(f(s))^{-1}(C(v))$, then $f(s)(\omega)\in C(v)$, $v\in f(s)(\omega)$.

Thus there is a $u\in U$, $u\in s(\omega)$ satisfying $f(u)=v$,

so $\omega\in s^{-1}(c(u))$, $\omega\in \bigcup_{f(u)=v} s^{-1}(c(u))$

Therefore,

$$(f(s))^{-1}(C(v))\subset \bigcup_{f(u)=v} s^{-1}(C(u))$$

On the contrary, suppose $\bigcup_{f(u)=v} s^{-1}(C(u))$, there is a $u\in U$ satisfying

$f(u)=v$, $\omega\in s^{-1}(C(u))$. Thus $s(\omega)\in C(u)$, $u\in s(\omega)$,

$f(u)=v\in f(s(\omega))$, $\omega\in(f(s))^{-1}(C(u))$

Therefore,

$$\bigcup_{f(u)=v} s^{-1}(C(u))\subset(f(s))^{-1}(C(V))$$

In summing up, we have

$$(f(s))^{-1}(C(u))=\bigcup_{f(u)=v} s^{-1}(C(u))$$

4. Intersection union and complement operations of random sets

Suppose $s, s_1, s_2\in S(\Omega.\mathcal{A}.P, U, \mathcal{B}, \check{\mathcal{B}})$.

Intersection $s_1\cap s_2$, union $s_1 \cup s_2$ of s_1, s_2 and complement s^c of s are defined as following:

$$\forall\,\omega\in\Omega,\quad (s_1\cap s_2)(\omega)\triangleq s_1(\omega)\cap s_2(\omega) \tag{4.1}$$

$$(s_1\cup s_2)(\omega)\triangleq s_1(\omega)\cup s_2(\omega) \tag{4.2}$$

$$(s^c)(\omega)\triangleq(s(\omega))^c \tag{4.3}$$

Obviously, $s_1\cap s_2\in S$, $s_1\cup s_2\in S$, $s^c\in S$.

Theorem 4.1 $(s_1\cap s_2)^{-1}(C(U)) = s_1^{-1}(C(u))\cap s_2^{-1}(C(u))$ (4.4)

$$(s_1\cup s_2)^{-1}(C(u)) = s_1^{-1}(C(u))\cup s_2^{-1}(C(u)) \tag{4.5}$$

$$(s^c)^{-1}(C(u)) = (s^{-1}(C(u)))^c \tag{4.6}$$

$$\mu_{\underset{\smile}{(s_1\cap s_2)}}(u) = P(s_1^{-1}(C(u))\cap s_2^{-1}(C(u))) \tag{4.7}$$

$$\mu_{(\underset{\vee}{s_1 \cup s_2})}(u) = P(s_1^{-1}(C(u)) \cup s_2^{-1}(C(u)) \tag{4.8}$$

$$\mu_{(\underset{\vee}{s^c})}(u) = 1-P(s^{-1}(C(u)) = \mu_{(\underset{\vee}{s})^c}(u) \tag{4.9}$$

The proof is obvious.

5. Relationship between the operations of random subsets and fuzzy subsets.

From definition of projectable fuzzy subset, we know that each random subset s on U correspond to a fuzzy subset $\underset{\sim}{A}$ on U so that $\underset{\sim}{A}$ is projectable fuzzy subset of s.

[2] , [3] studied contrary problems.

Given probability space $(\Omega, \mathcal{A}, P)$, relationship of operations of random subsets and fuzzy subsets are given by following theorem.

Theorem 5.1 Suppose that $s \in S(\Omega, \mathcal{A}, P, U, \mathcal{B}, \check{\mathcal{B}})$, $t \in S(\Omega, \mathcal{A}, P; V, \mathcal{B}_2, \check{\mathcal{B}}_2)$, $s_1, s_2 \in S(\Omega, \mathcal{A}, P, U, \mathcal{B}, \check{\mathcal{B}})$, and f is a mapping from U to V, then

$$(\underset{\vee}{s \times t}) \leq \underset{\vee}{s} \times \underset{\vee}{t} \tag{5.1}$$

$$\underset{\vee}{f(s)} \geqslant f(\underset{\vee}{s}) \tag{5.2}$$

$$(\underset{\vee}{s_1 \cap s_2}) \leq \underset{\vee}{s_1} \cap \underset{\vee}{s_2} \tag{5.3}$$

$$(\underset{\vee}{s^c}) = (\underset{\vee}{s})^c \tag{5.4}$$

$$(\underset{\vee}{s_1 \cup s_2}) \geqslant \underset{\vee}{s_1} \cup \underset{\vee}{s_2} \tag{5.5}$$

The proof is obvious.

6. Relation between the collection of bivariate random vectors and the collection of projrctable random intervals.

Suppose that R^2 is a real plane, $\mathcal{B}_0^2$ is a Borel field on R^2

$(\Omega, \mathcal{A}, P)$ is given probability space.

$$V \triangleq \{\xi \mid \xi : \Omega \to R^2, \xi \text{ is } \mathcal{A}\text{-}\mathcal{B}_0^2 \text{ measurable}\} \tag{6.1}$$

I is collection of all random intervals on R,

δ^{Ω} is collection of all mappings from Ω to δ.

Let $\psi : V \to \delta^{\Omega}$

$\forall \xi = (\xi_1, \xi_2) \in V$, $\psi(\xi)$ is defined as following

$\psi(\xi): \Omega \to \delta$

$$\forall \omega \in \Omega,\ \psi(\xi)\ (\omega) \triangleq [\min\{\xi_1(\omega), \xi_2(\omega)\},\ \max\{\xi_1(\omega), \xi_2(\omega)\}] \tag{6.2}$$

$$\lrcorner_x \triangleq (-\infty, x] \times [x, +\infty), \qquad x \in R \tag{6.3}$$

$$x_\Gamma = [x, +\infty) \times (-\infty, x], \qquad \forall x \in R \tag{6.4}$$

Theorem 6.1 ψ is a mapping from V to I and $\forall x \in R$

$$\psi(\xi)^{-1}(C(x)) = \xi^{-1}(\lrcorner x) \cup \xi^{-1}(x_\Gamma) \tag{6.5}$$

Write $s = \psi(\xi)$, we have:

$\forall x \in R$

$$\mu_s(x) = P(\{\omega \mid \xi(\omega) \in \lrcorner x) + P(\{\omega \mid \xi(\omega) \in x_\Gamma\}) - P(\{\omega \mid \xi(\omega) = (x,x)\}) \tag{6.6}$$

If ξ has distribution density $p(x_1, x_2)$, then

$$\begin{aligned}\mu_s(x) &= \iint_{x_\Gamma} p(x_1, x_2)dx_1dx_2 + \iint_{\lrcorner x} p(x_1, x_2)dx_1dx_2 \\ &= 1 - \int_{-\infty}^{x}\int_{-\infty}^{x} p(x_1, x_2)dx_1dx_2 - \int_{x}^{+\infty}\int_{x}^{+\infty} p(x_1, x_2)dx_1dx_2\end{aligned} \tag{6.7}$$

Proof: Writing $$R^2/ \triangleq \{(x, y) \in R^2 \mid x \le y\} \tag{6.8}$$

$$\mathcal{B}^2/ \triangleq \{D \mid D \in \mathcal{B}_0^2 \text{ and } D \subseteq R^2/\} \tag{6.9}$$

Suppose $\xi = (\xi_1, \xi_2) \in V$, then ξ is $\mathcal{A}$-$\mathcal{B}_0^2$ measurable,

thus ξ_1, ξ_2 are $\mathcal{A}$-$\mathcal{B}_0$ measurable.

Let $\eta_1 \triangleq \min\{\xi_1, \xi_2\}$, $\eta_2 \triangleq \max\{\xi_1, \xi_2\}$, $\eta \triangleq (\eta_1, \eta_2)$,

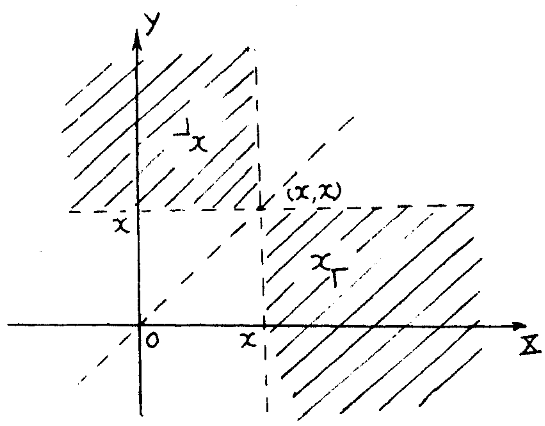
then η is a mapping from Ω to $R^2/$, and η is $\mathcal{A}-\mathcal{B}^2/$ measurable.

By theorem 4 of [1], $\psi(\xi) \in I$, that is, ψ is a mapping from V to I.

$$\begin{aligned}&\psi(\xi)^{-1}(C(x)) \\ &= \{\omega \mid x \in \psi(\xi)\ (\omega)\} \\ &= \{\omega \mid \min\{\xi_1(\omega), \xi_2(\omega)\} \le x \le \max\{\xi_1(\omega), \xi_2(\omega)\}\} \\ &= (\{\omega \mid \xi_1(\omega) \le x\} \cup \{\omega \mid \xi_2(\omega) \le x\}) \cap (\{\omega \mid \xi_1(\omega) \ge x\} \cup \{\omega \mid \xi_2(\omega) \ge x\}) \\ &= \{\omega \mid \xi(\omega) \in \lrcorner x\} \cup \{\omega \mid \xi(\omega) \in x_\Gamma\} \\ &= \xi^{-1}(\lrcorner x) \cup \xi^{-1}(x_\Gamma)\end{aligned}$$

The formula 6.5 is proved.

The formulas 6.6 and 6.7 and obvious from following figure.



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