PIONEERS OF FUZZINESS

Walenty Ostasiewicz
Academy of Economics, Wrocław, Komandorska 118/120

1. Introduction

Bibliographical survey of fuzziness given by Gaines and Kohout [5] is still the best one among many other contributions of this kind. The main objective of this paper is to draw an attention to some works not mentioned in [5] and to give a little different explanation of the idea of quasi-ontology proposed by T. Kubiński [8].

The problem of vagueness, impreciseness and ambiguity is a very old one, which absorbed an attention of many researchers. It has been discussed particularly widely by philosophers. The most eminent among them is undoubtedly *W. G. Leibniz* who has devoted entirely eleven chapters of his "Nouveaux Essais" to the discussion of words problem: their meaning, impreciseness, using abusing etc.

There were also undertaken some attempts to treat precisely the phenomenon of impreciseness. Roughly speaking, "precisely" means "mathematically", and mathematics in its turn means almost the same as theory of sets, an essential part of which forms the calculus of classes. And this calculus has been taken by many researchers as the point of departure for the construction a calculi of imprecise concepts.

The classical class calculus was developed in ancient times and it has been presented by Aristotle in his "Analytics" as a theory of syllogism. From the formal point of view this calculus is equivalent to a system of Boolean algebra, say, of characteristic functions introduced in 1936 by Ch. de la Vallee-Poussin. In the same year the notion of characteristic function has been generalized by E. Szpilrajn who extends it to the sequences of sets. This notion has been also generalized by H. Weyl in 1940, by A. Kaplan and H. F. Schott in 1951 and by L. Zadeh in

1965, in such a way that the range of the generalized characteristic function is the unit interval [0, 1] instead of two-elements set {0, 1}. All the three proposals are equivalent to a system of De Morgan algebra. And this system is just what is widespread as a suitable calculus for imprecise concepts.

Any Boolean algebra is also De Morgan algebra, so that one can say that De Morgan algebras generalize the classical calculus of classes.

The classical characteristic function has been generalized also by H. Rasiowa and R. Sikorski. Two elements set $B = \{0, 1\}$ of two logical values is replaced by any logical algebra A e.g. minimal logic, positive logic, intuitionistic logic, modal logic etc.

Generalized characteristic function $f: X \longrightarrow A$ "determines an A-subset X_f of X in the following sense: for every $x \in X$ the sentence x belongs to X_f has the logical value $f(x) \in A$" (see [15]).

A different generalization of classical calculus of classes represents Leśniewski's Ontology, which in its turn has been modified by T. Kubiński to quasi-ontology. Leśniewski's system rose from a mere antinomies which have shaken so badly the Cantorian conception of set theory.

In order to overcome Russell's paradox S. Leśniewski has developed his original system for the foundation of mathematics. His total system consists of three parts: *protothetics*, *ontology* and *mereology*.

Very briefly, protothetics is a generalized propositional calculus, mereology is a theory of sets in collective sense which deals with part-whole relationship rather than with element-membership relation, and ontology is a theory of names which deals with "the general properties of being".

As a ontology forms a basis for Kubiński's theory of imprecise names it will be briefly explain in this paper. It seems to be indispensable for understanding Kubiński's theory.

2. Weyl's calculi

Vagueness pertains mainly to the properties of real objects or phenomena. Identifying the notions "property" or "feature" with the notion "set", G. Boole has developed calculus known today as an Boolean algebra (see [1]).

Very often it is extremely difficult or even impossible to decide whether or not possesses an object a certain property. It seems to be much more natural to replace this all-or nothing principle by the continuous scale of grades $f(x)$ of the extent to which a given property is possessed by an object $x \in X$. The mapping $f: X \longrightarrow [0, 1]$ can be considered therefore as some kind of generalization of classical two-valued predicate. Such a vague or phenomenal predicates have been introduced for the first time by H. Weyl in 1940 (see [19]). He has defined following operations for such predicates:

$$f \cap g = \min(f, g), \qquad f \cup g = \max(f, g), \qquad \bar{f} = 1 - f.$$

Essentially the same calculus (De Morgan algebra) he has considered also within the framework of probability theory. In the work [19] there are discussed moreover two other, substantially different calculi.

One of them in todays terminology is called a Heyting algebra or Brouwer algebra (see [1, 3]) in which the properties are identified with (or interpreted as) an open sets of some topological space. These type of calculi turned to be very fruitful in considering modal logics, three-valued calculi of imprecise concepts, and in other branches.

The second calculus is an non-distributive logic of quantum mechanics originated in 1936 by Birkhoff and von Neumann which formally is equivalent to an orthocomplemented modular lattice.

Recently this calculus is often discussed within the framework of fuzzy sets theory. Unfortunately, it is far not clear neither from [19] nor from any of recently published papers how this logic corresponds to propositions containing vague names or (and) vague functors.

3. Kaplan-Schott calculus for empirical classes

The notion "class" in its scientific use is rather different from the same notion used in mathematics. Classical mathematics, as it is well-known, deals with an ideal entities, and if it is attended to be applied for description of real-world phenomena these latter have to be idealized.

Classical calculus of classes presented for example in "Principia Mathematica" as an idealization of "empirical"classes disregards the vagueness characterizing actual usage of the notion "class".

A. Kaplan and H. F. Schott have proposed a calculus which should provide a more adequate explication of classes in their scientific use than is afforded by the conventional calculus of classes. In contrast to classical calculus of classes their calculus has been called calculus for empirical classes (CEC).

The main features of this calculus are presented here with some, not essential, changes of terminology and notations with respect to original paper [7].

Let X be any non-empty (finite) set considered as a universe of discourse (field of inquiry in the original terminology). Suppose that L is so called feature space i.e. the set of all possible "vectors" (profils in the original terminology) $l = (l_1, l_2, \ldots, l_n)$ of characteristics which can be assigned any element $x \in X$. On the set L is supposed to be defined "normed" weighting function $w: L \longrightarrow [0, 1]$.

The values of this function are fixed by convention according to defined empirical class $\tilde{A}$ in X. They are called therefore a nominal probabilities and denoted by $w_{\tilde{A}}(l)$ which is read as "the weight from the profile l to the class $\tilde{A}$". Since a profile $l = l(l_1, \ldots, l_n)$ is assigned to an element $x \in X$, the notation $w_{\tilde{A}}(l_x)$ can be used, which will be simplified to $\tilde{A}(x)$, or even to $A(x)$. Using todays language, one can say therefore that empirical class A in X is a mapping $A: X \longrightarrow [0, 1]$ which is defined as a composition of two mappings: one from X to L, the other from L to [0, 1].

The basic operations for empirical classes are defined as follows:

$$(A \cup B)(x) = \max[A(x),\ B(x)] = A(x) + B(x) - (A \cap B)(x),$$
$$(A \cap B)(x) = \min[A(x),\ B(x)],$$
$$\bar{A}(x) = 1 - A(x),$$
$$(A - B)(x) = \max[A(x) - B(x),\ 0].$$

Partial ordering by class inclusion is defined as follows:

$$A \subset B \leftrightarrow A = A \cap B,$$

and the different ordering, called W-sharpening, is defined as follows:

$$A > B \leftrightarrow \begin{cases} A(x) \geq B(x), & \text{for } B(x) \geq 1/2 \\ B(x) \geq A(x), & \text{for } B(x) < 1/2 \end{cases}.$$

For any empirical class A its *idealization* iA is defined by

$$iA(x) = \begin{cases} 1, & \text{if } A(x) \geq 1/2 \\ 0, & \text{if } A(x) < 1/2 \end{cases}.$$

It has been stated (without proof) that nearly all the propositions in "Principia Mathematica" on classical calculus of classes hold for corresponding operations on empirical classes. "These include the laws of commutativity and associativity for sum and product, De Morgan's theorems, the laws of tautology, the two distribution laws, the law of double negation and of transposition, and the transitivity of inclusion" [7].

In order to have the analogous to some other propositions of classical calculus in [7] are defined particular classes:

1. Null class: $N(x) = 0$,
2. Universal class: $U(x) = 1$,
3. Quasi-null class: $nu(x) \leq 1/2$,
4. Quasi-universal class: $un(x) \geq 1/2$.

Suppose that $A = nu$ means that A is equal to some quasi-null class, then the classes defined above satisfy the following properties:

$$A \cup N = A \, ,$$
$$U \subset A \leftrightarrow U = A \, ,$$
$$A \cup \bar{A} = un \, ,$$
$$A \cap \bar{A} = nu \, ,$$
$$A \subset B \leftrightarrow A \cap \bar{B} = nu \, ,$$
$$A \subset B \leftrightarrow \bar{A} \cup B = un \, .$$

Having introduced the notion of membership to the degree p, symbolized by $\in^{P}$, the principle of extensionality has been defined by:

$$A = B \leftrightarrow (x \in^{P} A \leftrightarrow x \in^{P} B) \text{ for all } x \in X, \; p \in [0, 1].$$

It is very interesting to note the fact, though not explicitly expressed in [7], that classical calculus is an idealization of empirical calculus. Two terms are defined to have *the same* meaning, if the corresponding classes (extensions) are equal, and these terms have *substantially the same* meaning, if the idealizations of corresponding classes are equal.

Suppose that on set X the probability measure P is defined, then the net weight NW(A) of a class A (the probability of fuzzy event, in recent terminology) is defined as follows:

$$NW(A) = \sum_{x \in X} P(x) \cdot A(x) \, ,$$

and the quantity ρ defined by the formula:

$$\rho(A, B) = NW(A \cap B) \, / \, NW(A)$$

is called a correlation between two classes, for which the following assertions are true:

1. If $A \neq N$, then $\rho(A, B)$ is unique ,
2. $B \subset A \rightarrow \rho(A, B) = 1$,
3. $A \neq N \rightarrow \rho(A, B) \geq 0$,
4. $\rho(A, B \cup C) = \rho(A, B) + \rho(A, C) - \rho(A< B \cap C)$,
5. $\rho(A, B \cap C) + \rho(A, B) \cdot \rho(A \cap B, C)$.

Paper [7] contains also discussion about some applications of proposed calculus.

4. Leśniewski's Ontology

St.Leśniewski (1886 - 1939), together with J.Łukasiewicz (1878 - 1956) belong to the most famous disciples of K.Twardowski, they were founders of Warsaw logical school. Among their students the most known are A.Tarski, A.Lindenbaum, M.Presburger, J.Słupecki, B.Sobociński, and other.

The discovery of paradoxes provoked a whole series of attempts to reconstruct the foundations of mathematics. The most known are proposals given by E.Zermelo, B.Russell and L.E.J.Brouwer. Lesser known is deductive system developed by S.Leśniewski, which is superiors in many aspects to other proposals.

Unfortunately, some of Leśniewski's original manuscripts were destroyed in Warsaw Uprising. Many information about his system were reconstructed basing on the notes from Leśniewski's lectures made by his students.

Leśniewski's total system, as it was just mentioned, encompasses three parts: protothetics, ontology and mereology.

The most relevant to traditional theory of sets is an ontology. The first observations on ontology were made by Leśniewski in a course of lectures during the years 1919 - 1920 (see [10, 21]).

Ontology has been treated by Leśniewski himself as a theory of "general principles of being". Now, this theory is also known as "calculus of names".

All expressions of any language, S. Leśniewski divides into three semantic categories: names, sentences and functors, and the latter are divided into name-forming and sentence-forming functors. For example names "Kate" and "Agathy" using the name-forming functor "and" can be combined into the name "Kate and Agathy". Similarly, sentences "Kate is diligent" and "Agathy is lazy" using sentence-forming functor "and" can be combined into sentence "Kate is diligent and Agathy is lazy". In order to avoid any confusion there are used different symbols for the functors which are read equally. For example symbols $\wedge$, $\vee$, $-$, $\rightarrow$ are used for statement-forming functors: conjunction, alternative, negation and implication, and the corresponding name-forming functors are symbolized by letters **K**, **A**, **N** and **C**.

Very roughly speaking, ontology is a theory of expressions of the following type: "x is A". It is worth noticing that the English verbal form "is" has three different meanings in the following sentences: Socrates is a man, the dog is an animal, Socrates is the husband of Xanthippe (see [4, 18]). The same word "is" in these sentences denotes correspondingly: the membership-relation , the inclusion-relation, and the identity relation.

In Polish as in Latin the copula "jest" or "est", respectively, joins two nouns without any articles, so is also in sentences of ontology. As a formal counterpart of the natural language "is" , Leśniewski uses symbol "ε" as a sentence-forming functor, which is a solely primitive term in ontology and its meaning is determined by a single axiom of ontology.

It is very interesting that "Leśniewski is completely successful in providing through his axiom, his own term "ε" with a meaning which is a kind of conglomerate of the three mentioned meanings without falling into contradiction" [4].

The only axiom of ontology formulated by Leśniewski in 1920 is following:

A1. $x \varepsilon X \leftrightarrow \exists y(y \varepsilon x) \wedge \forall y,z(y \varepsilon x \wedge z \varepsilon x \rightarrow y \varepsilon z) \wedge \forall y(y \varepsilon x \rightarrow y \varepsilon X)$.

In words it means that Leśniewskian ε-sentence "$x \ \varepsilon \ X$" is true only if to the left of "ε" stands non-empty, singular name, and to the right any name.

For example sentences as "Pegasus is big" or "The dog is an animal" are treated as false, because in the first sentence "Pegasus" is an empty name and "dog" in the second sentence is a general (but not singular) name.

It is well known that exists a perfect analogy between the calculus of propositions and the algebra of sets. Similarly, elementary ontology as a calculus of names constructed over the narrower predicate calculus without identity is equivalent to a system of atomic Boolean algebra (see [6]).

5. Kubiński's quasi-ontology

Kubiński attempted to develop a formal theory about the real-world phenomena which would encompasses a vague terms. The notion "vagueness" is discussed in this paper both from pragmatic and semantic standpoints, and what is much more interesting, also from syntactic point of view. This latter approach suggested by L. Borkowski has never been considered in the literature before Kubiński. For that reason only syntactic definition of vagueness is considered in this note. According to this definition, a non-individual term b is called vague, if there exists an individual (or singular) term a such that neither the expression "a is b" nor the expression "a is non-b" are theses of definite systems called Quasi-ontologies.

Any quasi-ontology contains as its proper part a deductive system Ω. This system is based an two primitive terms: "ε" and "N". The first of them is Leśniewskian sentence-forming functor "epsilon", which corresponds to particle "is", and the second term "N" is name-forming functor which corresponds to verbal negation "non".

System Ω is based an three axioms and two definitions, which are presented here in verbal, intuitive form (original expressions are reproduced in [5]):

A1. Axiom of ontology (explained in previous section).

A2. If x is y, then x is-not non-y. (This is a definition of functor ND.

D1. Definition of functor A (for alternative name, like John or Mary).

D2. Definition of functor K (for conjunctive name, like Paul and Simon).

A3. De Morgan laws for names formed by functors A and K.

Kubiński has proved that system Ω is consistent, incomplete, and its axioms are independent.
Within this system one can prove for example, that if some object is red, then it is not non-red, but the contrary is not true.

Suppose that X is some non-empty set of names and there are fixed some rules of inference. Quasi-ontology associated with set X is defined as a system Ω augmented by some numbers of axioms of singular existence and by some number of empirical theses. Axiom of singular existence is an expression of the type "a is a", and means that exists exactly one a.

An empirical thesis is an expression of one of the following types: " a is b", "a is non-b", "neither a is b nor a is non-b", where a is a singular (individual) name, and b is non-singular name.

Accordingly to what set of empirical theses would be assumed and what particular axioms of singular existence would be chosen, one obtains a different quasi-ontologies.

The family of all possible quasi-ontologies associated with a given set X is denoted by Q(X). The elements of this family can be considered as a theories of real-world phenomenon and they can be partially ordered with respect to the number of imprecise terms.

Acknowledgement

I would like to thank Professors J. Drewniak, K. Haagen, P. Schönemann and Z. Świtalski for their advice, helpful conversation, encouragement and for providing the literature.

R E F E R E N C E S

1. G.Birkhoff: Lattice theory, Providence, Rhode Island, 1967.
2. M.Black: Vagueness, Phil. of Science, 4, 1937, 427-455.
3. M.B.Curry: Foundations of mathematical logic, McGraw-Hill, N.Y., 1963.
4. A.A.Fraenkel, Y.Bar-Hillel, A.Levy: Foundations of set theory, NH, 1973.
5. B.R.Gaines, L.J.Kohout: The fuzzy decade; a bibliography of fuzzy systems and closely related topics, in: M.M.Gupta G.N.Saridis, B.R.Gaines (eds.), Fuzzy automata and decision processes, North-Holland, Amsterdam, 1977, 403-490.
6. B.Iwanuś: On Leśniewsli's elementary ontology, Studia logica, 31, 1973, 73-119.
7. A.Kaplan, H.F.Schott: A calculus for empirical classes, Methodos, 1951, vol.III, N° 11, 165-188.
8. T.Kubiński: Nazwy nieostre, Studia logica, 7, 1958, 115-179.
9. T.Kubiński: An attempt to bring logic nearer to colloquial language, Studia logica, 10, 1960, 61-75.
10. C.Lejewski, S.Leśniewski (1886-1939): Wiadomości Mat., 28, 1990, 153-182.
11. S.Leśniewski: Podstawy ogólnej teorii mnogości, Moscow, 1916.
12. J.Łukasiewicz: Selected works, North-Holland and PWN, 1970.
13. K.Menger: Ensembles flous et fonctions aleatories, Comptes Rendus de l'Academie de Sciences Paris 232, 1951, 2001-2003.
14. W.Ostasiewicz: Still the same ghost of modality, Proc. of Polish Sym. on Fuzzy and Interval Math., Poznań (in print).
15. H.Rasiowa: A generalization of a formalized theory of fields of sets on non-classical logics, Rozprawy Matematyczne, 42, 1964, 3-29.
16. H.Rasiowa, R.Sikorski: A proof of the completeness theorem of Gödel, Fundamenta Mathematicae 37, 1950, 193-200.
17. E.Szpilrajn: Sur l'equvalence des suites d'ensembles of l'equivalence des fonctions, Fundamenta Mathematicae, 26, 1936, 302-326.

18. J.T.J.Strzednicki, V.F.Rickety, J.Czelakowski (ems.): Leśniewski's System Ontology and Mereology, Martinus Nijhoff and Ossolineum, 1984.
10. H.Weyl: The ghost of modality in: Philosophical essays in memory of Edmund Husserl, Cambridge (Mass.), 1940, 278-303.
20. H.Weyl: Mathematic and logic, Amer. Math. Month. 53, 1946, 2-13.
21. J.Woleński: Filozoficzna szkoła lwowsko-warszawska, PWN, Warszawa 1985.