PROBABILISTIC-QUASI-PSEUDO-METRIC-SPACES

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In this paper some results concerning the bitopological properties of probabilistic-quasi-pseudo-metric-spaces are presented.

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Probabilistic metric spaces were first introduced by Menger[6] as a generalization of the notion of a metric space in which the "distance" between any two points is a probability distribution function. These spaces are assumed to satisfy axioms which are quite similar to the axioms satisfied in an ordinary metric space.

Such probabilistic metric spaces have been studied by several authors, for example Wald [12], Schweizer and Sklar [9], [10].

Note that in [5] Kramosil and Michalek apply the concept of fuzziness to the classical notions of metric and metric spaces and compared this notion with that of probabilistic metric space, and both the conceptions are proved to be equivalent in certain sense.

It is well known that if we do not require that $d(x,y)=0$ implies $x=y$, the standard theorems on metric spaces are easy to generalize. However, if the symmetry of $d(x,y)$ is dropped, generalization of metric space results need not be obvious.

Willson [14] initited the study of those spaces, and several authors have contributed to the subsequent development of quasi-pseudo-metric spaces, for example Albert [1], Kelly [4], Reilly [8].

It is worthy of note that quasi-metrics are of some interest in topology and elsewhere. Some Quasi-metrics of biological origin have been recently studied by Waterman, Smith and Beyer [12]. Domiaty [2] has discussed the relevance of Quasi-metrics to the structure of space-time.

In the present paper we initiate the study of probabilistic-quasi-pseudo-metric-space as a common generalization of the concepts of quasi-pseudo-metric and probabilistic-metric spaces.

Our paper is divided into two parts, each devoted to one main topic. They are:

I Axioms of probabilistic-quasi-pseudo-metric-spaces with examples.

II The topology generated by a probabilistic-quasi-pseudo-metric and its properties.

1. Introduction. A **distance distribution function** (briefly d.d.f.) is a non-decreasing function $F:[0,+\infty]\to[0,1]$, which is left-continuous on $(0,+\infty)$ and takes on the values $F(0)=0$, and $F(+\infty)=1$. The set of all d.d.f's, denoted by Δ^+, is equiped with modified Levy metric d_L (see p.45 of [10]). The metric space (Δ^+, d_L) is compact, and hence complete, and Δ^+ is partially ordered by usual order for real-valued functions. Let u_a be the element of Δ^+ defined by

$$(1.1)\qquad u_a = \begin{cases} 1_{(a,+\infty]}, & \text{for all } a\in[0,+\infty), \\ 1_{\{+\infty\}}, & \text{for } a=+\infty . \end{cases}$$

A **triangle function** $*$ is defined to be a binary operation on Δ^+ which is non-decreasing in each component, and if $(\Delta^+,*)$ is an Abelian monoid with identity u_0. A triangle function $*$ is proper if

$$(1.2)\qquad u_{a+b} \leq u_a * u_b \quad \text{for all } a,b\in R^+ .$$

A triangle function $*$ is continuous if it is continuous with respect to the metric topology induced by d_L.

We also have the following results, which will by needed later:

2. LEMMA.[see.p.48 of [10]] For any $F\in\Delta^+$, and for any $t>0$,

$$(2.1)\qquad F(t) > 1-t \quad \text{iff} \quad d_L(F,u_0) < t .$$

From this we immediately obtein

3. <u>LEMMA</u>. If $F, G \in \Delta^+$ and $F \leq G$, then

(3.1) $$d_L(G, u_0) \leq d_L(F, u_0).$$

4. <u>DEFINITION</u>. A probabilistic-quasi-pseudo-metric-space(briefly, a p-q-p-metric-space) is a triple $(X,P,*)$ whre X is nonempty set, P is a function from X^2 into Δ^+, $*$ is a triangle function, and the following conditions are satisfied:

(4.1) $P(x,x) = u_0$, for all $x \in X$,

(4.2) $P_{xy} * P_{yz} \leq P_{xz}$, for all $x,y,z \in X$.

If P satisfies the condition

(4.3) $P_{xy} = u_0$ iff $x = y$, then $(X,P,*)$ is a probabilistic-quasi-metric-space (p-q-metric-space). If also

(4.4) $P_{xy} = P_{yx}$ and $P_{xy} \neq u_0$, if $x \neq y$, then $(X,P,*)$ is a probabilistic-metric-space.

5. <u>DEFINITION</u>. Let $(X,P,*)$ be a p-q-p-metric-space and let the function $Q: X^2 \to \Delta^+$ be defined by

(5.1) $Q(x,y) = P(y,x)$, for all $x,y \in X$, then a triple $(X,Q,*)$ is also p-q-p-metric-space. We say that P and Q are conjugate and denote the set X with this structure by $(X,P,Q,*)$.

6. <u>EXAMPLE</u>. Let X be a partially ordered set, the function $P: X^2 \to \Delta^+$ be defined by

(6.1) $$P(x,y) = \begin{cases} u_0 & \text{if } x \leq y, \\ G & \text{if } x \nleq y, \end{cases}$$ where $G \in \Delta^+$, then the triple $(X,P,+)$ is a p-q-p-metric-space for all triangle functions $*$.

7. <u>EXAMPLE</u>. Let X be a partially ordered set and the function $P: X^2 - \Delta^+$ be defined by

(7.1) $$P(x,y) = \begin{cases} u_0 & \text{if } x = y, \\ F & \text{if } x < y, \\ G & \text{if } x \nleq y, \end{cases}$$ where $F \geq G \in \Delta^+$, then the triple $(X,P,*)$ is a p-q-metric-space forall $*$.

8. **EXAMPLE**. Let X be a partially ordered set and let $(X,F,*)$ be a probabilistic-metric-space such that:

(8.1) $F(x,y) \geqslant G$ for all $x,y \in X$, where $G \in \Delta^+$, the Function P: $X^2 \to \Delta^+$ is given by

$$(8.2) \quad P(x,y) = \begin{cases} F(x,y) & \text{if } x \leq y, \\ G & \text{if } x \not\leq y, \end{cases}$$ then $(X,P,*)$ is a p-q-metric space.

9. **EXAMPLE**. If (X,p) is a quasi-pseudo-metric-space and the function $P:X^2 \to \Delta^+$ is defined by

(9.1) $P(x,y) = \varepsilon_{p(x,y)}$ for $x,y \in X$, and $*$ is such that

(9.2) $\varepsilon_a * \varepsilon_b = \varepsilon_{a+b}$ for all $a,b \in R^+$, then $(X,P,*)$ is a proper

p-q-p-metric space. In the other direction, if $(X,P,*)$ is a proper p-q-p-metric-space, and there is a function $p:X^2 \to R^+$ such that(10.1) holds, then (X,p) is a quasi-pseudo-metric-space.Consequently, the class of quasi-pseudo-metric-spaces may be viewed as a subclass of the class of proper p-q-p-metric spaces.

10. **EXAMPLE**. Let $(\Omega,\mathcal{A},\mu)$ be a probability space,(Y,p) be a quasi-pseudo-metric space and X a set of functions from Ω into Y.Let function $P:X^2 \to \Delta^+$ be such that for all $x,y \in X$ and $t>0$, the set $\{w \in \Omega: p(x(w),y(w)) < t\} \in \mathcal{A}$ and $P_{xy}(t) = \mu\{w \in \Omega : p(x(w),y(w)) < t\}$,then (X,P) is a p-q-p-metric-space under triangle function $*_W$ givenbby

(10.1) $F *_W G(t) = \sup[\mathrm{Max}(F(t_1)+G(t_2)-1,0), t_1+t_2 = t]$, $F,G \in \Delta^+$

Proof. We need only establish (4.2). For any $x,y,z \in X$ and $t>0$, let $t_1,t_2 > 0$ be such that $t_1+t_2 = t$. Define the sets A,B,C by $A = \{w \in \Omega: p(x(w),y(w)) < t_1\}$, $B = \{w \in \Omega: p(y(w),z(w)) < t_2\}$, $C = \{w \in \Omega: p(x(w),z(w)) < t$ .Since p satisfies the triangle inequality,it follows that $A \cap B \subseteq C$. Hence $\mu(C) \geqslant \mu(A \cap B) \geqslant \mu(A) + \mu(B) - 1$. Thus $P_{xz}(t) \geqslant \sup[\mathrm{Max}(P_{xy}(t_1) + P_{yz}(t_2) - 1,0), t_1 + t_2 = t]$.

11. **EXAMPLE**. Fuzzy quasi-pseudo-metric-spaces.The 3-tuple (X,M,T) is fuzzy quasi-pseudo-metric space if X is an arbitrary set, T is a t-norm, and M is a fuzzy set $X^2 \times [0,+\infty]$ for all $x,y,z \in X$ satisfying the following conditions:

(11.1) $M(x,y,0) = 0$ and $M(x,y,+\infty) = 1$,

(11.2) $M(x,x,t) = 1$ for all $t>0$,

(11.3) $M(x,y,\cdot):[0,+\infty]\to[0,1]$ is left-continuous;

(11.4) $M(x,z,t+s) \geq T(M(x,y,t),M(y,z,s))$, $t,s>0$.

12. LEMMA. $M(x,y,\cdot)$ is non-decreasing for all $x,y\in X$.

Proof. Suppose $0<t<s$. Then by (11.4) and (11.2) $M(x,y,s)\geq T(M(x,x,s-t),M(x,y,t)) = T(1,M(x,y,t)) = M(x,y,t)$.

From this we immediately obtain

13. THEOREM. Let T be acontinuous t-norm and (X,M,T) be fuzzy quasi-pseudo-metric space. Then the triple $(X,P,*_T)$ is a p-q-p-metric space, where function $P:X^2\to\Delta^+$ is given by formula: $P(x,y) = M(x,y,\cdot)$ and triangle function $*_T$ is given by $F *_T G(t) = \sup[\, T(F(t_1),G(t_2)),\ t_1+t_2 = t \text{ and } t_1,t_2>0\,]$.

Consequently, the class of fuzzy quasi-pseudo-metric spaces with continuous t-norms may be viewed as a subclass of the class of probabilistic-quasi-pseudo-metric-spaces.

TOPOLOGY

14. In the theory of probabilistic-metric spaces, the concept of a neighbourhood can be introduced and defined with the aid of the probabilistic distance function [9]. We apply a similar procedure in the theory of p-q-p-metric-spaces. It is the topic of this section.

15. DEFINITION. Let $(X,P,*)$ be a p-q-p-metric-space. For any $x\in X$ and $t>0$, the P-neighbourhood of x is the set

(15.1) $N_x^P(t) = \{\, y\in X:\ d_L(P_{xy},u_0)<t \,\}$.

16. THEOREM. Let $(X,P,*)$ be a p-q-p-metric-space. If $*$ is continuous, then the collection of all P-neighbourhoods form a base for the topology on X.

Proof. For all $t>0$, $x\in N_x^P(t)$ since $d_L(P_{xx},u_0) = d_L(u_0,u_0) = 0$. If $t_1<t_2$, then $N_x^P(t_1)\subseteq N_x^P(t_2)$ by (15.1) since $N_x^P(\min(t_1,t_2))\subseteq N_x^P(t_1)\cap N_x^P(t_2)$. And let $y\in N_x^P(t)$ and $d_L(P_{xy},u_0)=k$. Then $t-k>0$, and uniform continuity of $*$ implies that there exists a $t'>0$ such that $d_L(P_{xy}*G,\ P_{xy})<t-k$ whenever $d_L(G,u_0)<t'$.
Now let $z\in N_y^P(t)$. Then $d_L(P_{yz},u_0)<t'$ and by (5.2),(3.1) $d_L(P_{xz},u_0)\leq$

$d_L(P_{xy} * P_{yz}, u_o) \leq d_L(P_{xy} * P_{yz}, P_{xy}) + d_L(P_{xy}, u_o) < t - k + k = t.$
Hence $z \in N_x^P(t)$. Thus $N_y^P(t) \subseteq N_x^P(t)$.

The topology determined in this way by P will be denoted by T_P. Similarly, Q determined a topology T_Q on X. Thus the natural topological structure associated with a p-q-p-metric on a set X is that of the set with two topologies and denoted by (X, T_P, T_Q).

17. DEFINITION([3]). The bitopological space (X, T_1, T_2) is pairwise semi-Hausdorff if for each pair of distinct points $x, y \in X$ there is a T_1-open set U and T_2-open set V disjoint from U such that either $x \in U$, $y \in V$ or $x \in V$, $y \in U$.

18. DEFINITION([13]). (X, T_1, T_2) is pairwise Hausdorff if for each pair of distinct points $x, y \in X$ there is a T_1-open set U and T_2-open set V disjoint from U such that $x \in U$, and $y \in V$.

19. THEOREM. Let $(X, P, *)$ be a p-q-p-metric space such that for all $x \neq y$ implies $P_{xy} \neq u_o$ or $Q_{xy} \neq u_o$, then the bitopological space (X, T_P, T_Q) is pairwise semi-Hausdorff. If in addition (4.3) holds, then (X, T_P, T_Q) is pairwise Hausdorff.

Proof. If $x \neq y$, then $P_{xy} \neq u_o$ or $Q_{xy} \neq u_o$. We assume that $P_{xy} \neq u_o$, whence $0 < k = d_L(P_{xy}, u_o)$. By the uniform continuity of $*$, there exists a $t > 0$ such that $d_L(G_1 * G_2, u_o) < k$, whenever $d_L(G_1, u_o) < t$ and $d_L(G_2, u_o) < t$. Suppose $z \in N_x^P(t) \cap N_y^Q(t)$. Then $d_L(P_{xz}, u_o) < t$ and $d_L(Q_{yz}, u_o) < t$, whence (4.2) and (3.1) yield $d_L(P_{xy}, u_o) \leq$ $d_L(P_{xz} * P_{zy}, u_o) = d_L(P_{xz} * Q_{yz}, u_o) < k$, an impossibility. Thus $N_x^P(t) \cap N_y^Q(t)$ is empty.

The second part follows easily from the (4.3). Indeed, if $x \neq y$, then $P_{xy} \neq u_o$ and $Q_{xy} \neq u_o$.

20. DEFINITION([4]). Let (X, T_1, T_2) be a bitopological space. Then T_1 is regular with respect to T_2 if for each point $x \in X$ and each T_1-closed set P such that $x \in P$ there is a T_1-open set U and T_2-open set V disjoint from U that $x \in U$ and P V.

21. DEFINITION([4]). (X, T_1, T_2) is pairwise normal if for each T_1-closed set A and T_2-closed set B disjoint from A there is a T_1-open set V containing B and T_2-open set U disjoint from V containing A.

22. DEFINITION. A triangle function $*$ is sup continuous if for each family $F_l \to \Delta^+ : l \in L$ and $G \in \Delta^+$ $\sup[(F_l * G : l \in L] = [\sup F_l : l \in L] * G.$

23. **LEMMA**. Let $(X,P,*)$ be a p-q-p-metric-space, $A \ X$, and let $*$ be sup continuous, and $P_{xA}=\sup[(P_{xy}: y\in A)]$, then $P_{xA} \geq P_{xz} * P_{zA}$, for $z \in X$.

24. **LEMMA**. Let $(X,P,*)$ be a p-q-p-metric-space, $A \ X$, and $*$ be sup continuous, then the function $f_A: X \to [0,1]$ given by $f_A = d_L(P_{xA}, u_o)$ is Q-upper semi-continuous. If in addition $* \geq *_w$ (10.1), then the function f_A is P-lower semi-continuous. Similarly, the function $g_A = d_L(Q_{xA}, u_o)$ is P-upper semi-continuous and Q-lower semi-continuous.

Proof. The proof depends on showing that for every $t \in R^+$, the sets: $V = \{y \in X: d_L(P_{yA}, u_o) < t\}$, $U = \{y \in X: d_L(P_{yA}, u_o) < t\}$, are respectively Q-open and P-closed.

1° Let $z \in V$. Since $d_L(P_{zA}, u_o) = a < t$, $t - a > 0$, by uniform continuity of $*$ there is $t' > 0$ such that $d_L(G * P_{zA}, P_{zA}) < t - a$, whenever $d_L(G, u_o) < t'$, and now, $r \in N_z^Q(t')$ since $d_L(Q_{zr}, u_o) = d_L(P_{rz}, u_o) < t'$ and $d_L(P_{rA}, u_o) \leq d_L(P_{rz} * P_{zA}) + d_L(P_{zA}, u_o) < t - a + a = t$, therefore $N_z^Q(t') \subseteq V$, since V is Q-open.

2° Now we show that $U = \overline{U}^P$. Let $h > 0$ and by (9.1) we have $P_{zr} * P_{rA}(t+h) + h \geq \mathrm{Max}[P_{zr}(h) + P_{rA}(t) - 1, 0] + h \geq 1 - h + P_{rA} - 1 + h = P_{rA}$ if $r \in N_z^P(h)$, on the other hand $P_{rA}(t+h) + h \geq \mathrm{Min}[P_{zr}(h), P_{rA}(t)] \geq P_{zr} * P_{rA}(t+h) \geq P_{zr} * P_{rA}(t)$, since for all $h > 0$ and $r \in N_z^P(h)$

$$d_L(P_{zr} * P_{rA}, P_{rA}) < h. \tag{24.1}$$

Now, let $z \in \overline{U}^P$ and $z \in U$. Since $d_L(P_{zA}, u_o) = b < t$, $b - t > 0$ and for all $h < b - t$ by (24.1) we have $b = d_L(P_{zA}, u_o) \leq d_L(P_{zr} * P_{rA}, u_o) \leq d_L(P_{zr} * P_{rA}) + d_L(P_{rA}, u_o) < b - t + d_L(P_{rA}, u_o)$, since $d_L(P_{rA}, u_o) > t$ if $r \in N_z^P(h)$ and $U \cap N_z^P(h) = \emptyset$, therefore $U = \overline{U}^P$.

25. **THEOREM**. Let $(X,P,*)$ be a p-q-p-metric space, where triangle function $+$ is sup continuous such that $+ \ +_w$ (10,1). Then the bitopological space (X, T_P, T_Q) is pairwise regular and pairwise normal.

Proof. By Lemma 24 for every $x \in X$ and each $t > 0$, the set $y \in X$: $d_L(P_{xy}, u_o) < t$ is Q-closed and each point $x \in X$ has a P-neighbourhood base of Q-closed sets. Hence T_P is regular with respect to T_Q Simlilarly, T_Q is regular with respect to T_P.

Let A and B be disjoint subsets of X such that A is P-closed and B is Q-closed, then by Lemma 24

$A = \{x \in X: d_L(P_{xA}, u_o) = 0\}$ and $B = \{x \in X: d_L(Q_{xB}, u_o) = 0\}$

Let $U = \{x \in X: d_L(P_{xA}, u_o) < d_L(Q_{xB}, u_o)\}$, $V = \{x \in X: d_L(Q_{xB}, u_o) < d_L(P_{xA}, u_o)\}$. Then $U \cap V = \emptyset$, $A \subseteq U$ and $B \subseteq V$, and we can establish pairwise normality by showing that U and V are respectively Q-open and P-open. We show that V is P-open. Suppose that $x_o \in V$ and $d_L(P_{x_oA}, u_o) - d_L(Q_{x_oB}, u_o) = k > 0$. By Lemma 24 $d_L(P_{xA}, u_o)$ is P-lower semi-continuous and $d_L(Q_{xB}, u_o)$ is P-upper semi-continuous, since there exists $t' > 0$ such that if $x \in N^P_{x_o}(t')$, then $d_L(P_{xA}, u_o) > d_L(P_{x_oA}, u_o) + \frac{k}{4}$ and $d_L(Q_{x_oB}, u_o) > d_L(Q_{xB}, u_o) + \frac{k}{4}$. Thus $d_L(P_{xA}, u_o) - d_L(Q_{xB}, u_o) - [d_L(P_{x_oA}, u_o) - d_L(Q_{x_oB}, u_o)] + \frac{k}{2} > \frac{k}{2} > 0$, and so $x \in V$, that $N^P_{x_o}(t') \subseteq V$.

26. <u>COROLLARY</u>([4]). Let (X,p) be a quasi-pseudo-metric-space. Then the bitopological space (X, T_p, T_q) is pairwise regular and normal.

Proof. This follows from Theorem 25 and from the fact that every quasi-pseudo-metric-space (X,p) is p-q-p-metric-space $(X,P,+)$ given by (9.1) and the topologies of those spaces are equivalent.

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