# PROBABILITY OF FUZZY INTUITIONISTIC SETS

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ABSTRACT. The notion of probability measure for fuzzy intuitionistic sets is introduced and some of its properties are described. Also, the notion of a fuzzy intuitionistic random variable and its basic characteristics are presented.

Key words: Fuzzy sets, fuzzy intuitionistic sets, probability of fuzzy intuitionistic fuzzy sets, fuzzy random variable.

## INTRODUCTION

K. Atanasov and S. Stoeva have recently introduced the notion of a fuzzy intuitionistic set [2,3,5] as a generalization of the notion of Zadeh's fuzzy set [10]. There were published articles concerning problems of lattice fuzzy intuitionistic sets [6] and fuzzy intuitionistic relations [1]. Then there were disscused fuzzy intuitionistic programs [4] and the problem of a correlation between fuzzy intuitionistic sets [7]. The next article [9] describes some concepts of measuring the fuzziness and the nonfuzziness of such sets.

In our article we introduce the concept of probability for fuzzy intuitionistic set theory such that, in the case when a fuzzy intuitionistic set is an ordinary fuzzy set, the introduced probability reduces to the fuzzy probability measure proposed by Zadeh in [11] which is - as it seems - the simplest (see [8]).

To simplify our considerations, we assume in this paper that the space $E$ is finite, i.e. $E = \{x_1, x_2, \ldots, x_n\}$. The extension to more general spaces is possible and it is now being worked on by the authors.

## 1. PRELIMINARIES

By an intuitionistic set [5] in the universum $E$ we mean the structure $A = \{(x, \mu_A(x), \nu_A(x)): x \in E\}$ where the functions $\mu_A, \nu_A : E \to [0,1]$ are such that $0 \le \mu_A(x) + \nu_A(x) \le 1$ and define, respectively, degrees of the belonging (the function $\mu_A$) and the non-belonging (the function $\nu_A$) of the given element $x$ from $E$ to the fuzzy intuitionistic set $A$.

An example of such a set is disscussed in [3]. The basic relations and operations on fuzzy intuitionistic sets are defined as follows:

$$A \subset B \iff \mu_A(x) \le \mu_B(x) \text{ and } \nu_A(x) \ge \nu_B(x), \ \forall x \in E, \tag{1.1}$$

$$A = B \iff A \subset B \text{ and } B \subset A, \tag{1.2}$$

$$A \cap B = \{(x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)): x \in E\} \text{ where}$$

$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x) \quad \text{and} \quad \nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x), \quad \forall x \in E, \tag{1.3}$$

$$A \cup B = \{(x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)) : x \in E\} \quad \text{where}$$

$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x) \quad \text{and} \quad \nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x), \quad \forall x \in E, \tag{1.4}$$

$$A' = \{(x, \mu_{A'}(x), \nu_{A'}(x)) : x \in E\} \quad \text{where}$$

$$\mu_{A'}(x) = \nu_A(x) \quad \text{and} \quad \nu_{A'}(x) = \mu_A(x), \quad \forall x \in E, \tag{1.5}$$

$$A \cdot B = \{(x, \mu_{A\cdot B}(x), \nu_{A\cdot B}(x)) : x \in E\} \quad \text{where}$$

$$\mu_{A\cdot B}(x) = \mu_A(x) \cdot \mu_B(x) \quad \text{and} \quad \nu_{A\cdot B}(x) = \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x), \quad \forall x \in E, \tag{1.6}$$

$$A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x)) : x \in E\} \quad \text{where}$$

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \quad \text{and} \quad \nu_{A+B}(x) = \nu_A(x) \cdot \nu_B(x), \quad \forall x \in E; \tag{1.7}$$

the symbol $\wedge$ stands for the minimum, and for the maximum. It can easily be checked that the operations $\cap$, $\cup$, $\cdot$, $'$, $+$, are commutative, associative and satisfy the laws of the Morgan. An ordinary fuzzy set written down in the convention of an intuitionistic set is the structure $A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in E\}$. The empty fuzzy intuitionistic set $\emptyset = \{(x,0,1) : x \in E\}$ and the universum $E = \{(x,1,0)\}$. Moreover, two fuzzy intuitionistic sets $A$ and $B$ are called disjoint when $A \cap B = \emptyset$.

## 2. THE NOTION OF PROBABILITY

Let in the universum $E = \{x_1, x_2, \ldots, x_n\}$ a probability measure $P = \{p_1, p_2, \ldots, p_n\}$ be defined such that $p_i \geq 0 \ \forall i$ and $\sum_{i=1}^{n} p_i = 1$

and $p_i = p(\{x_i\})$. Let us define in $E$ any fuzzy intuitionistic set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$ treated as an intuitionistic event.

DEFINITION. By a probability of a fuzzy intuitionistic event in the universum $E$ with a probability measure $P$ defined we mean the number

$$\tilde{P}(A) = \sum_{i=1}^{n} \frac{1}{2} (\mu_A(x_i) + 1 - \nu_A(x_i)) p_i . \tag{2.1}$$

Let us notice that, when $A$ is a Zadeh fuzzy set, formula (2.1) reduces to

$$P(A) = \sum_{i=1}^{n} \mu_A(x_i) p_i , \tag{2.2}$$

which expresses the probability of a fuzzy event defined in [11] by Zadeh.

For probability measure (2.1) introduced above, the following properties (which constitute Kolmogorov axiom system) hold:

A1) $\tilde{P}(\emptyset) = 0,$ (2.3)

A2) $\tilde{P}(E) = 1,$ (2.4)

A3) $A \cap B = \emptyset \Rightarrow \tilde{P}(A \cup B) = \tilde{P}(A) + \tilde{P}(B).$ (2.5)

For example, we shall prove property A3. We have

$$\tilde{P}(A \cup B) = \sum_{i=1}^{n} \frac{1}{2} ( \mu_{A \cup B}(x_i) + 1 - \nu_{A \cup B}(x_i)) p_i =$$

$$= \sum_{i=1}^{n} \frac{1}{2} [ (\mu_A(x_i) \quad \mu_B(x_i)) + 1 - (\nu_A(x_i) \wedge \nu_B(x_i)) ] p_i =$$

$$= \sum_{i=1}^{n} \frac{1}{2}[(\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i) \wedge \mu_B(x_i)) + 1 - (\nu_A(x_i) +$$

$$+ \nu_B(x_i) - \nu_A(x_i) \quad \nu_B(x_i))]p_i =$$

$$+ \sum_{i=1}^{n} \frac{1}{2}[(\mu_A(x_i) + \mu_B(x_i) + 0) + 1 - (\nu_A(x_i) + \nu_B(x_i) - 1)]p_i =$$

$$= \sum_{i=1}^{n} \frac{1}{2}[(\mu_A(x_i) - \nu_A(x_i) + 1) + (\mu_B(x_i) - \nu_B(x_i) + 1)]p_i =$$

$$= \sum_{i=1}^{n} \frac{1}{2}(\mu_A(x_i) + 1 - \nu_A(x_i))p_i + \sum_{i=1}^{n} \frac{1}{2}(\mu_B(x_i) + 1 - \nu_B(x_i))p_i =$$

$$= \tilde{P}(A) + \tilde{P}(B).$$

We use here the remark on the fuzzy intuitionistic empty set and the property which states that

$$a \vee b = a + b - a \wedge b \quad \text{for all numbers } a \text{ and } b.$$

It can also be shown that

$$\tilde{P}(A') = 1 - \tilde{P}(A), \tag{2.6}$$

$$A \subset B \Rightarrow \tilde{P}(A) \leq \tilde{P}(B), \tag{2.7}$$

$$\tilde{P}(A \cup B) = \tilde{P}(A) + \tilde{P}(B) - \tilde{P}(A \cap B). \tag{2.8}$$

## 3. INDEPENDENCE OF EVENTS

Let two fuzzy intuitionistic events $A$ and $B$ be given in some finite spaces $E_1$ and $E_2$, respectively. Let a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ be defined in $E_1$ and a probability distribution $Q = \{q_1, q_2, \ldots, q_m\}$ in $E_2$. Let us consider a combined occurrence of the events A and B in the universum $E = E_1 \times E_2$ in which the probability distribution is given by $p_{ij}$ where $p_{ij} = p_i \cdot q_j$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$.

DEFINITION. We say that events A and B are independent in E when $\tilde{P}(A \cdot B) = \tilde{P}(A) \cdot \tilde{P}(B)$ where $A \cdot B$ is the intuitionistic fuzzy set described by (1.6).

The above definition is analogous to that of independence for ordinary fuzzy events, introduced in [11]. It can be proved that

T1) Any fuzzy intuitionistic event A and the universum E are always independent as well as any fuzzy intuitionistic event A and the fuzzy intuitionistic empty set $\emptyset$.

T2) If intuitionistic events A,C and B,C are independent and $A \cap B = \emptyset$, then the events $A \cup B$ and C are independent too.

In consequence, we have the following

DEFINITION. By the conditional probability of a fuzzy intuitionistic event A, given B, we mean

$$\tilde{P}(A|B) = \frac{\tilde{P}(A \cdot B)}{\tilde{P}(B)} . \tag{3.1}$$

If A and B are independent, then $\tilde{P}(A|B) = \tilde{P}(A)$.

The theorems below are valid.

THE THEOREM ON TOTAL PROBABILITY. If $B_1, B_2, \ldots, B_k$ are pairwise disjoint fuzzy intuitionistic events such that $B_1 \cup B_2 \cup \ldots \cup B_k = E$ and $\tilde{P}(B_i) > 0$ for $i = 1,2,\ldots,k$, then, for any event A in E, we have

$$\tilde{P}(A) = \tilde{P}(B_1) \cdot \tilde{P}(A|B_1) + \ldots + \tilde{P}(B_k) \cdot \tilde{P}(A|B_k). \tag{3.2}$$

THE BAYES THEOREM. If $A_1, A_2, \ldots, A_k$ are pairwise disjoint fuzzy intuitionistic events such that $A_1 \cup A_2 \cup \ldots \cup A_k = E$ and $B_1, B_2, \ldots, B_m$ are also pairwise disjoint fuzzy intuitionistic events such that $B_1 \cup B_2 \cup \ldots \cup B_m = E$ and $\tilde{P}(A_i) > 0$ for $i = 1,2,\ldots,k$ and $\tilde{P}(B_j) > 0$ for $j = 1,2,\ldots,m$, then, for all $i$ and $j$,

$$\tilde{P}(B_j|A_i) = \frac{\tilde{P}(B_j) \cdot \tilde{P}(A_i|B_j)}{\tilde{P}(B_1) \cdot \tilde{P}(A_i|B_i) + \ldots + \tilde{P}(B_m) \cdot \tilde{P}(A_i|B_m)} \tag{3.3}$$

holds.

## 4. FUZZY INTUITIONISTIC RANDOM VARIABLE

DEFINITION. If, to any fuzzy intuitionistic event $A$ in $E = \{x_1, x_2, \ldots, x_n\}$, we assign any real number, then such a function will be called a fuzzy intuitionistic random variable for the event A.

Any set of pairs $\{(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \cdot x_i, p_i),\ i=1,2,\ldots,n\}$ will be called a distribution of a fuzzy intuitionistic random variable A, where the number $x_i$ is its value, and $p_i = p(\{x_i\})$.

By an expected value of such a variable we mean the number

$$E_n(X) = \sum_{i=1}^{n} \frac{1}{2}(\mu_A(x_i) + 1 - \nu_A(x_i)) \cdot x_i \cdot p_i \tag{4.1}$$

with the properties

1) $E_A(cX) = c \cdot E_A(X)$, $c$ = constant (4.2)

2) $E_A(I) = \tilde{P}(A)$ (4.3)

where $I$ is the function identically 1,

3) $E_A(X + Y) = E_A(X) + E_A(Y)$ (4.4)

where $X$ and $Y$ are fuzzy intuitionistic random variables for the same event $A$.

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