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THE CONDITIONAL EXPECTATIONS WITH RESPECT TO λ-ADDITIVE MEASURES

Li Lushu
Department of Mathematics, Huaiyin Teacher's College
Jiangsu Province, P.R.China

ABSTRACT

In this paper, we introduce the concepts of conditional expectations of fuzzy events with respect to λ-additive measures given a σ-subalgebra and a mapping, respectively. Some properties of them are investigated.

Keywords: λ-additive measure, λ-integral, conditional λ-expectation.

1. Preliminaries

Throughout this paper, let X be a non-empty basic set, $\mathbf{A}$ be a σ-algebra of subsets of X, and $\lambda > -1$. We will denote by $\mathbf{F}(\mathbf{A})$ the class of $\mathbf{A}$-measurable functions from X to [0, 1], and the elements of $\mathbf{F}(\mathbf{A})$ are called fuzzy events by Zadeh [4]. A set function u_λ: $\mathbf{A} \longrightarrow [0, 1]$ is said to be a λ-additive measure on $(X, \mathbf{A})$[1], if it satisfies $u_\lambda(\emptyset) = 0$, $u_\lambda(X) = 1$ and

$$\text{(i)}\ \{A_n, n \in \mathbf{N}\} \subseteq \mathbf{A},\ A_n \uparrow (\downarrow) A \Longrightarrow \lim_{n \to \infty} u_\lambda(A_n) = u_\lambda(A) \tag{1.1}$$

$$\text{(ii)}\ A, B \in \mathbf{A},\ A \cap B = \emptyset \Longrightarrow u_\lambda(A \cup B) = u_\lambda(A) + u_\lambda(B) + \lambda \cdot u_\lambda(A) \cdot u_\lambda(B) \tag{1.2}$$

The triplet $(X, \mathbf{A}, u_\lambda)$ will be called a λ-additive measure space. We say that two fuzzy events h_1, h_2 are u_λ-equialent, written as $h_1 = h_2$ u_λ-a.e, if there exists $E \in \mathbf{A}$, such that $\{x \in X: h_1 \neq h_2\} \subseteq E$ and $u_\lambda(E) = 0$.

Obviously, if u_λ is a λ-additive measure, then $u^* = \log_{1+\lambda}(1 + \lambda u_\lambda)$ is a probability measure[3]. Thus we can use the u^*-integral to define the λ-integral of fuzzy events $f \in \mathbf{F}(\mathbf{A})$ over $A \in \mathbf{A}$ as

$$\int_A f du_\lambda = \lambda^{-1} \cdot \left[(1+\lambda)^{\int_A f du^*} - 1 \right], \quad \forall f \in \mathbf{F}(\mathbf{A}),\ A \in \mathbf{A} \tag{1.3}$$

The λ-integral of f over X, $E_\lambda(f) = \int_X f du_\lambda$, is called λ-expectation of f. In order to introduce the conditional λ-expectations of fuzzy events, we may prove the following Radon-Nikodym-like theorem:

Theorem 1.1. Let $(X, \mathbf{A}, u_\lambda)$ be a λ-additive measure space. If $v_\lambda : \mathbf{A} \longrightarrow [0, 1]$ is a set function satisfying (1.1), (1.2) and $v_\lambda \ll u_\lambda$, i.e.

$$u_\lambda(A) = 0 \ (\forall A \in \mathbf{A}) \Longrightarrow v_\lambda(A) = 0 \tag{1.4}$$

Then there exists a unique fuzzy event $h \in \mathbf{F}(\mathbf{A})$ in the sense of u_λ-equivalence such that $v_\lambda(A) = \int_A h du_\lambda$ $(\forall A \in \mathbf{A})$. (cf.[8])

2. Conditional λ-expectation Given a σ-subalgebra

Let $(X, \mathbf{A}, u_\lambda)$ be a λ-additive measure space, $\mathbf{B}$ be a σ-subalgebra of $\mathbf{A}$. If $f \in \mathbf{F}(\mathbf{A})$, then $v_\lambda(A) = \int_A f du_\lambda$ $(\forall A \in \mathbf{A})$ is a set function $\mathbf{A} \longrightarrow [0, 1]$ satisfying (1.1), (1.2) and (1.4). If we restrict u_λ and v_λ to $\mathbf{B}$, then it follows from theorem 1.1 that there exists a unique fuzzy event $h \in \mathbf{F}(\mathbf{B})$ in the sense of u_λ-equivalence on $\mathbf{B}$ such that $\int_B f du_\lambda = \int_B h du_\lambda$ $(\forall B \in \mathbf{B})$.

Definition 2.1. Let $(X, \mathbf{A}, u_\lambda)$ be a λ-additive measure space and $\mathbf{B}$ be a σ-subalgebra of $\mathbf{A}$. For any $f \in \mathbf{F}(\mathbf{A})$, the conditional λ-expectation of f given $\mathbf{B}$, written as $E_\lambda(f|\mathbf{B})$, is a $\mathbf{B}$-measurable function defined up to a u_λ-equivalence on $\mathbf{B}$ by

$$\int_B E_\lambda(f|\mathbf{B}) du_\lambda = \int_B f du_\lambda, \ \forall B \in \mathbf{B} \tag{2.1}$$

The conditional λ-expectation defined above has the following properties:

Theorem 2.1. Suppose $f, f_n \in \mathbf{F}(\mathbf{A})$, $n \geq 1$. $\mathbf{B}$ is a σ-subalgebra of $\mathbf{A}$. We have:

(1) $E_\lambda(E_\lambda(f|\mathbf{B})) = E_\lambda(f)$

(2) If $f \in \mathbf{F}(\mathbf{B})$ or $\mathbf{B} = \mathbf{A}$, then $E_\lambda(f|\mathbf{B}) = f$ u_λ-a.e.

(3) If $\mathbf{B} = \{\emptyset, X\}$, then $E_\lambda(f|\mathbf{B}) = E_\lambda(f)$ u_λ-a.e.

(4) If $f = c$ u_λ-a.e., then $E_\lambda(f|\mathbf{B}) = c$ u_λ-a.e.

(5) If $f_1 + f_2 \in \mathbf{F}(\mathbf{A})$, then $E_\lambda(f_1+f_2|\mathbf{B}) = E_\lambda(f_1|\mathbf{B}) + E_\lambda(f_2|\mathbf{B}) + \lambda E_\lambda(f_1|\mathbf{B}) E_\lambda(f_2|\mathbf{B})$

(6) If $f_n \uparrow(\downarrow) f$ u_λ-a.e., then $E_\lambda(f_n|\mathbf{B}) \uparrow(\downarrow) E_\lambda(f|\mathbf{B})$ u_λ-a.e.

Proof. Straightforward.

Theorem 2.2. (Smoothing Properties)

(1) On every non-null atom $B \in \mathbf{B}$ (i.e. $u_\lambda(B) > 0$ and if $C \in \mathbf{B}$, $C \subseteq B$ then $C = B$ or $C = \emptyset$), $E_\lambda(f|\mathbf{B})$ is a constant, the average of values of f on B, i.e.

$$E_\lambda(f|\mathbf{B}) = \frac{1}{u_\lambda(B)} \cdot \int_B f du_\lambda \tag{2.2}$$

(2) If $f \in \mathbf{F}(\mathbf{B})$, $g \in \mathbf{F}(\mathbf{A})$, then

$$E_\lambda(fg|\mathbf{B}) = f.E_\lambda(g|\mathbf{B}) \quad u_\lambda\text{-a.e.} \tag{2.3}$$

(3) If $\mathbf{B}_1 \subseteq \mathbf{B}_2$ are two σ-subalgebras of $\mathbf{A}$, then

$$E_\lambda[E_\lambda(f|\mathbf{B}_1)|\mathbf{B}_2] = E_\lambda[E_\lambda(f|\mathbf{B}_2)|\mathbf{B}_1] = E_\lambda(f|\mathbf{B}_1) \quad u_\lambda\text{-a.e.} \tag{2.4}$$

Proof. (1) If we set $B_0 = \{x \in X: E_\lambda(f|\mathbf{B})(x) = E_\lambda(f|\mathbf{B})(x_0)\} \cap B$, where $x_0 \in B$, then it follows from the fact that $B_0 = B$ because B is an atom that $E_\lambda(f|\mathbf{B})$ is a constant. In addition, as $E_\lambda(f|\mathbf{B}).u_\lambda(B) = \int_B E_\lambda(f|\mathbf{B})du_\lambda = \int_B f du_\lambda$ and $u_\lambda(B) > 0$, (3.3) holds at once.

(2) For every $A, B \in \mathbf{B}$, we have $\int_A E_\lambda(I_B.g|\mathbf{B})du_\lambda = \int_A (I_B.g)du_\lambda = \int_{A \cap B} g du_\lambda = \int_{A \cap B} E_\lambda(g|\mathbf{B})du_\lambda = \int_A I_B.E_\lambda(g|\mathbf{B})du_\lambda$, where I_B is the indicator function of B. Therefore (2.3) holds for the simple functions $f_n \in \mathbf{F}(\mathbf{B})$. By theorem 2.1 (6), we know that the assertion holds for any fuzzy event $f = \lim_{n \to \infty} f_n \in \mathbf{F}(\mathbf{B})$.

(3) Since $\mathbf{B}_1 \subseteq \mathbf{B}_2$ implies that $E_\lambda(f|\mathbf{B}_1)$ is $\mathbf{B}_2$-measurable, we have from (2.3) that $E_\lambda[E_\lambda(f|\mathbf{B}_1)|\mathbf{B}_2] = E_\lambda(f|\mathbf{B}_1).E_\lambda(1|\mathbf{B}_2) = E_\lambda(f\ \mathbf{B}_1)$ u_λ-a.e. In addition, forevery $A \in \mathbf{B}_1 \subseteq \mathbf{B}_2$, we have $\int_A E_\lambda[E_\lambda(f|\mathbf{B}_2)|\mathbf{B}_1)du_\lambda = \int_A E_\lambda(f|\mathbf{B}_2)du_\lambda = \int_A f du_\lambda = \int_A E_\lambda(f|\mathbf{B}_1)du_\lambda$. Hence $E_\lambda[E_\lambda(f|\mathbf{B}_2)|\mathbf{B}_1] = E_\lambda(f|\mathbf{B}_1)$ u_λ-a.e. Therefore (2.4) holds, and we finsh the proof of theorem 2.2.

3. Conditional λ-expectation Given a Mapping

Let $(X, \mathbf{A}, u_\lambda)$ be a λ-additive measure space, and T be a mapping from X to X'. If we set $\mathbf{A}' = \{A': A' \subseteq X', T^{-1}(A') \in \mathbf{A}\}$, then it is easily to prove that $\mathbf{A}'$ is a σ-algebra on X' which makes that T is a measurable mapping from $(X, \mathbf{A})$ to $(X', \mathbf{A}')$, and $\sigma(T) = T^{-1}(\mathbf{A}')$ is a σ-subalgebra of $\mathbb{A}$. By definition 2.1, we know that $E_\lambda(f|\sigma(T))$, the conditional λ-expectation of f given $\sigma(T)$, is a $\sigma(T)$-measurable function.

Definition 3.1. We write $E_\lambda(f|T)$ instead of $E(f|\sigma(T))$, and call it the conditional λ-expectation of fuzzy event f given T.

Theorem 3.1. Let $(X, \mathbb{A}, u_\lambda)$ be a λ-additive measure space, and T be a measurable mapping from $(X, \mathbb{A})$ to $(X', \mathbb{A}')$. If φ is a $\mathbb{A}'$-measurable function from X' to $[0, 1]$, and we define $u_\lambda^T(A') = u_\lambda(T^{-1}(A'))$, $\forall A' \in \mathbb{A}'$, then

(1) u_λ^T is a λ-additive measure on $(X', \mathbb{A}')$

(2) $$\int_{A'} \varphi du_\lambda^T = \int_{T^{-1}(A')} \varphi \circ T du_\lambda \ , \ \forall A' \in \mathbb{A}' \tag{3.1}$$

We may call u_λ^T the λ-additive measure induced by T on $(X', \mathbb{A}')$.

Proof. The frist assertion is straightforward. To prove the second assertion, we have $(u_\lambda^T)^*(A') = \log_{1+\lambda}(1+\lambda u_\lambda^T(A')) = u^*(T^{-1}(A'))$ for $\forall A' \in \mathbb{A}'$, and by the transformatiom theorem of u^*-integrals, we get $\int_{A'} \varphi du_\lambda^T =$

$$\lambda^{-1}[(1+\lambda)^{\int_{A'}\varphi d(u_\lambda^T)^*} - 1] = \lambda^{-1}[(1+\lambda)^{\int_{T^{-1}(A')}\varphi\circ T du^*} - 1] = \int_{T^{-1}(A')} \varphi\circ T du_\lambda$$

i.e. (3.1) holds, the second assertion is proved.

Theorem 3.2. Let $(X, \mathbb{A}, u_\lambda)$ be a λ-additive measure space, and $f \in \mathbb{F}(\mathbb{A})$. If T is a measurable mapping from $(X, \mathbb{A})$ to $(X', \mathbb{A}')$, then

$$E_\lambda(f|T) = \varphi\circ T \quad u_\lambda\text{-a.e.} \quad \text{on } \sigma(T) \tag{3.2}$$

Where φ is a $\mathbb{A}'$-measurable function from X' to $[0, 1]$, defined up to a u_λ-equivalence by $\int_{A'} \varphi du_\lambda^T = \int_{T^{-1}(A')} f du_\lambda$, $\forall A' \in \mathbb{A}'$.

Proof. We define $v_\lambda : \mathbb{A} \longrightarrow [0, 1]$ and $v_\lambda' : \mathbb{A}' \longrightarrow [0, 1]$ by $v_\lambda(A) = \int_A f du_\lambda$, $\forall A \in \mathbb{A}$; $v_\lambda'(A') = v_\lambda(T^{-1}(A'))$, $\forall A' \in \mathbb{A}'$, respectively. Then v_λ' satisfies (1.2) and (1.4). Applying theorem 1.1 to u_λ^T and v_λ', we get a $\mathbb{A}'$-measurable function φ from X' to $[0, 1]$, defined up to a u_λ^T-equivalence by

$$\int_{A'} \varphi du_\lambda^T = v_\lambda'(A') = \int_{T^{-1}(A')} f du_\lambda \ , \ \forall A' \in \mathbb{A}' \tag{3.3}$$

By the definition of $E_\lambda(f|T)$, we have

$$\int_{T^{-1}(A')} f du_\lambda = \int_{T^{-1}(A')} E_\lambda(f|T) du_\lambda, \quad \forall A' \in \mathbf{A}' \tag{3.4}$$

It follows, upon applying (3.1), (3.3) and (3.4), that

$$\int_{T^{-1}(A')} E_\lambda(f|T) du_\lambda = \int_{A'} \varphi du_\lambda^T = \int_{T^{-1}(A')} \varphi \cdot T du_\lambda, \ \forall A' \in \mathbf{A}'$$

Hence (3.2) holds, and the assertion is proved.

On account of the foregoing theorem, we see that $E_\lambda(f|T)$ is a function of the mapping T. As defined, $E_\lambda(f|T)$ is a $\sigma(T)$-measurable function on the original space $(X, \mathbf{A}, u_\lambda)$. Hence both interpretations are possible: either $E_\lambda(f|T)$ is considered as a function of the mapping T with values $E_\lambda(f|T)(y)$ for $y = T(x)$, or it is considered as a function on X with values $E_\lambda(f|T)(x)$ for $x \in X$, defined up to a u_λ^T- or u_λ-equivalence, respectively. We prefer to consider $E_\lambda(f|T)$ as the function on the origrnal space.

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