Transformation Theorems for Fuzzy Pan-integrals

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Abstract

In papers [1] and [2] we gave the definition of fuzzy pan-integral, discussed its elementary properties and proved some convergence theorems of sequence of fuzzy pan-integrals. In this paper, some transformation theorems for fuzzy pan-integrals are proved. These results reveal the relations between the fuzzy pan-integrals on fuzzy sets and the pan-integrals on classical sets.

§1. Definition of Fuzzy Pan-integral

Throughout this paper, let X be a nonempty set and $\mathscr{F}(X)=\{\underset{\sim}{A}: \underset{\sim}{A}:X\longrightarrow [0,1]\}$ be the class of all fuzzy subsets of X. Also we adopt the conventions: $0\cdot\infty=0$, $\inf_{t\in\emptyset}\{a_t;\ a_t\in[0,\infty]\}=\infty$.

Definition 1.1 [1] A nonempty subset $\underset{\sim}{\mathscr{F}}$ of $\mathscr{F}(X)$ is called a fuzzy σ-algebra if the following conditions are satisfied:

(1). $\emptyset$, $X\in\underset{\sim}{\mathscr{F}}$;

(2). if $\underset{\sim}{A}\in\underset{\sim}{\mathscr{F}}$, then $\underset{\sim}{A}^c\in\underset{\sim}{\mathscr{F}}$;

(3). if $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathscr{F}}$, then $\bigcup_{n=1}^{\infty}\underset{\sim}{A_n}\in\underset{\sim}{\mathscr{F}}$.

Definition 1.2 [1] Suppose that $\underset{\sim}{\mathscr{F}}$ is a fuzzy σ-algebra. A function $\underset{\sim}{\mu}: \underset{\sim}{\mathscr{F}}\longrightarrow\overline{R}_+$ is called a fuzzy measure, if it satisfies the following conditions:

(1). $\underset{\sim}{\mu}(\emptyset)=0$;

(2). for any $\underset{\sim}{A}$, $\underset{\sim}{B}\in\underset{\sim}{\mathscr{F}}$, if $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $\underset{\sim}{\mu}(\underset{\sim}{A})\leq\underset{\sim}{\mu}(\underset{\sim}{B})$;

(3). whenever $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathscr{F}}$, $\underset{\sim}{A_n}\subset\underset{\sim}{A_{n+1}}$, $n=1,2,\cdots$, then $\underset{\sim}{\mu}(\bigcup_{n=1}^{\infty}\underset{\sim}{A_n})=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A_n})$;

(4). whenever $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathscr{F}}$, $\underset{\sim}{A_n}\supset\underset{\sim}{A_{n+1}}$, $n=1,2,\cdots$, and there exists n_0, such that $\mu(\underset{\sim}{A_{n_0}})<\infty$, then $\mu(\bigcap_{n=1}^{\infty}\underset{\sim}{A_n})=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A_n})$.

Definition 1.3 [1] A mapping $f: X\longrightarrow(-\infty,\infty)$ is called a measurable function on $\underset{\sim}{\mathscr{F}}$, if for any $\alpha\in[-\infty,\infty]$, we have $F_\alpha=\{x: f(x)\geq\alpha\}\in\underset{\sim}{\mathscr{F}}$.

Denote the set of all measurable functions on $\underset{\sim}{\mathscr{F}}$ by $\underset{\sim}{M}$, and write $\underset{\sim}{M}^+=\{f: f\in\underset{\sim}{M}$ and $f\geq 0\}$.

Definition 1.4 [1] Let $(X,\underset{\sim}{\mathscr{F}})$ be a fuzzy measurable space, $(\overline{R}_+,\oplus,\odot)$ be an ordered commutative semi-ring, $\mathscr{B}=\{E: E\in\underset{\sim}{\mathscr{F}}$ and E is a classical set$\}$ (It is clear that $\mathscr{B}$ is a classical σ-algebra contained in $\underset{\sim}{\mathscr{F}}$). Then

(1). If I is the unit element of $(\overline{R}_+, \oplus, \odot)$, the extended real-valued function on X:

$$\chi_E(x)=\begin{cases} I & \text{if } x\in E \\ 0 & \text{if } x\bar{\in} E \end{cases}$$

is called the characteristic function of E, where $E\in\mathcal{B}$;

(2). The real-valued function s(x):

$$s(x)=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i}(x))$$

is said to be a simple function on X, if $0\leq\alpha_i<\infty$, $E_i\in\mathcal{B}$, $i=1,2,\cdots n$, and if $\alpha_i\neq\alpha_j$, $E_i\cap E_j=\emptyset$, $i\neq j$; $\bigcup_{i=1}^{n}E_i=X$.

We denote the set of all simple functions on X by S.

Definition 1.5 [1] Let $(X, \mathcal{F}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ be a fuzzy pan-space, $\underset{\sim}{A}\in\mathcal{F}$. For any $s(x)=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i}(x))\in S$,

we write $P_{\underset{\sim}{A}}(s)=\bigoplus_{i=1}^{n}(\alpha_i\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap E_i))$.

When $f\in\underset{\sim}{M}^+$, the fuzzy pan-integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $(p)\int_{\underset{\sim}{A}} f\, d\underset{\sim}{\mu}\triangleq\sup_{s\in s(f)}P_{\underset{\sim}{A}}(s)$, where $s(f)=\{s:\ s\in S,\ 0\leq s\leq f\}$.

§2. Transformation Theorems for Fuzzy Pan-integrals

Theorem 2.1 [1]. For any given $\underset{\sim}{A}\in\mathcal{F}$, if we define $\mu^*(E)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap E)$ for any $E\in\mathcal{B}$, then μ^* is a fuzzy measure on $\mathcal{B}$. It is called the fuzzy measure induced by $\underset{\sim}{\mu}$ and $\underset{\sim}{A}$.

From this theorem and definition 1.5, we can obtain the following result:

Transformation Theorem I. Let $\underset{\sim}{A}\in\mathcal{F}$, $f\in\underset{\sim}{M}^+$, and μ^* be the fuzzy measure defined in Theorem 2.1. Then $(p)\int_{\underset{\sim}{A}\cap E} f\, d\underset{\sim}{\mu}=(p)\int_E f\, d\mu^*$, whenever $E\in\mathcal{B}$. Particularly, $(p)\int_{\underset{\sim}{A}} f\, d\underset{\sim}{\mu}=(p)\int_X f\, d\mu^*$, where $(p)\int_E f\, d\mu^*$ is the pan-integral given in [3].

Proof. For any $E\in\mathcal{B}$, we have

$$(p)\int_{\underset{\sim}{A}\cap E} f\, d\underset{\sim}{\mu}=\sup_{s\in s(f)}\left(\bigoplus_{i=1}^{n}(\alpha_i\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap E\cap E_i))\right)=\sup_{s\in s(f)}\left(\bigoplus_{i=1}^{n}(\alpha_i\odot\mu^*(E\cap E_i))\right)$$

$$=(p)\int_E f\, d\mu^*.$$ The theorem is proved.

Definition 2.1 Let $f\in\underset{\sim}{M}^+$, f is called a homeomorphic function, if f is a bijection and $f(E)\in\mathcal{B}_+$ whenever $E\in\mathcal{B}$, where $\mathcal{B}_+$ is the class of all Borel-sets on R_+.

Theorem 2.2 Let $\underset{\sim}{A}\in\mathcal{F}$, $f\in\underset{\sim}{M}^+$. If we define $\mu(B)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap f^{-1}(B))$ for any $B\in\mathcal{B}_+$, then μ is a fuzzy measure on $(R_+, \mathcal{B}_+)$.

It is very easy to prove Theorem 2.2 according to the definition of the fuzzy measure.

Transformation Theorem II. Let $(X, \mathcal{F}, \underset{\sim}{\mu})$ be a fuzzy measure space, $\underset{\sim}{A} \in \mathcal{F}$, $f \in \underset{\sim}{M}^+$ be a homeomorphic function, μ be the fuzzy measure defined in Theorem 2.2 and $g: R_+ \to R_+$ be a Borel function. Then

$(p)\int_{\underset{\sim}{A}} g\circ f d\underset{\sim}{\mu} = (p)\int_{R_+} g\, d\mu$. Where $g\circ f$ is the composition of f and g and $(p)\int_{R_+} g\, d\mu$ is the pan-integral of g on R_+ with respect to μ.

Proof. We denote $s(g\circ f)=\{s: s \leq g\circ f, s \in S\}$. $\bar{s}(g)=\{\bar{s}: \bar{s} \leq g$ and $\bar{s}$ is a nonnegative simple function on $(R_+, \mathcal{B}_+)\}$. Obviously $g\circ f \in \underset{\sim}{M}^+$.

If $s(x)=\bigoplus_{i=1}^{n} (\alpha_i \odot \chi_{E_i}(x)) \in s(g\circ f)$, we take $\bar{s}=\sum_{i=1}^{n}\alpha_i f(E_i)$, then $\bar{s} \in \bar{s}(g)$. So we have $P_{\underset{\sim}{A}}(s)=\bigoplus_{i=1}^{n} (\alpha_i \odot \underset{\sim}{\mu}(\underset{\sim}{A} \cap E_i)) = \bigoplus_{i=1}^{n} (\alpha_i \odot \underset{\sim}{\mu}(\underset{\sim}{A} \cap f^{-1}(f(E_i))))$

$= \bigoplus_{i=1}^{n} (\alpha_i \odot \mu(f(E_i))) = P_{R_+}(\bar{s}) \leq \sup_{\bar{s}\in \bar{s}(g)} P_{R_+}(\bar{s}) = (p)\int_{R_+} g\, d\mu$. and therefore $(p)\int_{\underset{\sim}{A}} g\circ f\, d\underset{\sim}{\mu} \leq (p)\int_{R_+} g\, d\mu$.

On the other hand, if $\bar{s}=\sum_{j=1}^{m} \beta_j B_j \in \bar{s}(g)$, we take

$s(x)=\bigoplus_{j=1}^{m} (\beta_j \odot \chi_{f^{-1}(B_j)}(x))$, obviously $s \in s(g\circ f)$. Thus we have

$$P_{R_+}(\bar{s}) = \bigoplus_{j=1}^{m} (\beta_j \odot \mu(B_j)) = \bigoplus_{j=1}^{m} (\beta_j \odot \underset{\sim}{\mu}(\underset{\sim}{A} \cap f^{-1}(B_j)))$$

$$= P_{\underset{\sim}{A}}(s) \leq \sup_{s \in s(g\circ f)} P_{\underset{\sim}{A}}(s) = (p)\int_{\underset{\sim}{A}} g\circ f\, d\underset{\sim}{\mu}.$$

It follows that $(p)\int_{R_+} g\, d\mu \leq (p)\int_{\underset{\sim}{A}} g\circ f\, d\underset{\sim}{\mu}$. The theorem is proved.

References

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