SEQUENCES OF EXTENDED FUZZY NUMBERS

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Abstract: In this paper, a family of extended fuzzy numbers have been introduced by using some equivalent relation; The spaces of convergent and bounded sequences of extended fuzzy number have been discussed.

Keywords: Convex compact set, Quotient set, Extended fuzzy number, Sequence of extended fuzzy numbers, Convergent and bounded sequences of extended fuzzy numbers.

1. Introduction

Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2] where it was shown that every convergent sequences is bounded. The spaces of bounded and convergent sequences of fuzzy numbers where introduced by S. Nanda [1] where it was shown that they are completed metric spaces. In this paper, we introduced a family of extended fuzzy numbers and obtain the all conclusions in [1]. [2].

2. Preliminaries

Let D denote the family of all convex compact sets on the k-dimensional Euclidean space R^k. For A, B $\in$ D define

$$A_i^- = \inf \{ t_i \mid (t_1, \ldots, t_i, \ldots t_k) \in A \}$$

$$A_i^+ = \sup \{ t_i \mid (t_1, \ldots, t_i, \ldots t_k) \in A \}, \quad i=1,2,\ldots,k.$$

$A \sim B$ iff $A_i^- = B_i^-$ and $A_i^+ = B_i^+$, $i=1,2,\ldots,k$.

It is easy to say that "$\sim$" defines an equivalent relation on D .

Based on it, we can determine the quotient set denoted by $D/\sim$.
For $A, B \in D/\sim$. define

$$A \le B \text{ iff } A_i^- \le B_i^- \text{ and } A_i^+ \le B_i^+ . \ i=1,\dots, k.$$

$$A = B \text{ iff } A \le B \text{ and } A \ge B$$

$$d(A, B) = \max_k (|A_i^- - B_i^-|, |A_i^+ - B_i^+|) = \bigvee_{i=1}^{k} (|A_i^- - B_i^-| \vee |A_i^+ - B_i^+|)$$

where " $\vee$ " is boolean sum. It is easy to prove that d defines a metric on $D/\sim$, $(D/\sim, d)$ is a complete metric space, and " $\le$ " is a partial order in $D/\sim$.

A k-dimensional fuzzy number is a fuzzy subset of R^k which is closed, convex, bounded and normal. Let $F(R^k)$ denote the set of all k-dimensional fuzzy numbers, for $x \in F(R^k)$ define X_λ as the following

$$X = \begin{cases} \{t \mid t \in R^k, \ x(t) \ge \lambda\} & \text{if } \lambda \in (0, 1] \\ \{t \mid t \in R^k, \ x(t) > 0\} & \text{if } \lambda = 0 \end{cases}$$

Let $L(R^k)$ denote the set of all k-dimensional fuzzy numbers which have compact support. In this words, if $x \in L(R^k)$, then for every $\lambda \in [0, 1]$, X_λ is convex and compact.

For $X, Y \in L(R^k)$, define

$$X \sim Y \text{ iff } X_\lambda \sim Y_\lambda \text{ for every } \lambda \in [0, 1]$$

It is easy to see that " $\sim$ " defines an equivalent relation on $L(R^k)$, the quotient set is denoted by $L(R^k)/\sim$.

For $X, Y \in L(R^k)/\sim$, define

$$X \le Y \text{ iff } X_\lambda \le Y_\lambda \text{ for every } \lambda \in [0, 1]$$

$$X = Y \text{ iff } X \le Y \text{ and } Y \le X$$

then " $\le$ " is a partial order in $L(R^k)/\sim$.

A subset $E/\sim$ of $L(R^k)/\sim$ is said to be bounded above if there exists $Z \in L(R^k)/\sim$, called an upper bounded of $E/\sim$, such that $X \le Z$ for every $x \in E/\sim$, A lower bound is defined similarly, $E/\sim$ is said to be bounded if it is both bounded above and bounded below.

Definition2.1 $L(R^k)/\sim$ is called a family of extended fuzzy num-

bers, i.e. x is called an extended fuzzy number if $x \in L(R^k)/\sim$.
Define a map $\bar{d}$: $(L(R^k)/\sim , L(R^k)/\sim) \to R^1$ by

$$\bar{d}(X, Y) = \sup_{0 \le \lambda \le 1} d(X_\lambda , Y_\lambda)$$

It is easy to see that d has determinate meaning.

Definition 2.2 A sequence $x = \{x_n\}_{n=1}^{\infty}$ of extended fuzzy numbers is said to be convergent to extended fuzzy number x_0, written as $\lim_n x_n = x_0$, if for every $\varepsilon > 0$ there exists positive integer n such that $\bar{d}(x_n, x_0) \le \varepsilon$ for $n \geq n_0$

Let C_0 denote the all convergent sequences of extended fuzzy numbers.

A sequence $x = \{x_n\}_{n=1}^{\infty}$ of extended fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that $\bar{d}(x_m, x_n) \le \varepsilon$ for $m, n \geq n_0$.

Let C_a denote the all Cauchy sequences of extended fuzzy numbers.

Definition 2.4 A sequence $x = \{x_n\}_{n=1}^{\infty}$ of extended fuzzy numbers is said to be bounded if the set $\{x_n \mid n \in N\}$ of extended fuzzy numbers is bounded.

Let B_0 denote the set of all bounded sequence of extended fuzzy numbers.

We now introduce the l^p space $(1 \le p < \infty)$ of extended fuzzy numbers as the following

$$l^p = \left\{ x = \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} \bar{d}(x_n, 0)^p < \infty \right\}$$

where the " 0 " is defined by $x(t) = \begin{cases} 1 & \text{if } t=(t_1 \ldots t_k)=(0, \ldots ,0) \\ 0 & \text{otherwise} \end{cases}$

3 . The results

We have the following results

Theorem 1 Every convergent sequence of extended fuzzy numbers is bounded , i.e. $C_0 \subset B_0$.

Proof: Let $x = \{x_n\}_{n=1}^{\infty} \in C_0$, $\lim_n x_n = x_0 \in L(R^k)/\sim$ then there exists n_0

such that $\bar{d}(x_n, x_0) \leq 1$ for $n \geq n_0$, it implies that

$$|((x_n)_\lambda)_i^- - ((x_0)_\lambda)_i^-| \leq 1 , \qquad |((x_n)_\lambda)_i^+ - ((x_0)_\lambda)_i^+| \leq 1$$

for every $\lambda \in [0, 1]$, $1 \leq i \leq k$ and $n \geq n_0$.

Put $\lambda = 0$

$$\alpha = \left\{\left[\bigvee_{i=1}^{k} (1+((x_0)_\lambda^+)_i)\right] \vee \left[\bigvee_{j=1}^{n_0} \left(\bigvee_{i=1}^{k} ((x_0)_\lambda^+)_i\right)\right]\right\} \in R^1$$

$$= \left\{\left[\bigwedge_{i=0}^{k}(-1+((x_0)_\lambda^-)_i)\right] \wedge \left[\bigwedge_{j=1}^{n_0} \left(\bigwedge_{i=1}^{k}((x_j)_\lambda^-)_i\right]\right\} \in R^1$$

then $z < x < y$ for every $n \in N$ and $x, y \in L(R^k)/\sim$

where $y = \begin{cases} 1 & \text{if } (t_1, \ldots, t_k) = (\quad , \ldots, \quad) \\ 0 & \text{otherwise} \end{cases}$

$z = \begin{cases} 1 & \text{if } (t_1, \ldots, t_k) = (\quad , \ldots, \quad) \\ 0 & \text{otherwise} \end{cases}$

that is to say $\{x_n\}_{n=1}^{\infty}$ is bounded.

Theorem 2 $\bar{d}$ defines a metric on $L(R^k)/\sim$ and $(L(R^k)/\sim , \bar{d})$ is a complete metric space.

Proof It is clear that $\bar{d}(x, y) \geq 0$, $\bar{d}(x, y) = 0$ if and only if $x = y$ and $\bar{d}(x, y) = \bar{d}(y, x)$ for arbitrary $x, y \in L(R^k)/\sim$. Notice that $(D/\sim, d)$ is a metric space, we have

$$d(x_\lambda, z_\lambda) \leq d(x_\lambda, y_\lambda) + d(y_\lambda, z_\lambda)$$

for arbitrary $x, y, z \in L(R^k)/\sim$ and $\lambda \in [0, 1]$, it implies that $\bar{d}(X, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$, so $(L(R^k)/\sim , \bar{d})$ is a metric space.

If $\{x_n\}$ is a Cauchy sequence in $L(R^k)/\sim$, then $\{(x_n)_\lambda\}$ is a Cauchy sequence in $D/\sim$ for each λ, $\lambda \in [0, 1]$ but $(D/\sim , d)$ is complete then $\lim_n (x_n)_\lambda = x_\lambda$, Now $\lim_n x_n = x$ and $x \in L(R^k)/\sim$, this proves the completeness of $L(R^k)/\sim$.

Theorem 3 C_0 is a complete metric space with the metric defined by

$$\rho(x, y) = \sup_n \bar{d}(x_n, y_n)$$

where $x=\{x_n\}_{n=1}^{\infty} \in C_0$ and $y=\{y_n\}_{n=1}^{\infty} \in C_0$.

Proof . It is clear that ρ is a metric on C_0, To show that C_0 is complete in this metric, let $\{x^{(i)}\}_{i=1}^{\infty}$ be a Cauchy sequence in C_0 . Then for each fixed n, $\{x_n^{(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence in $L(R^k)/\sim$. But $(L(R^k)/\sim , d)$ is complete, hence $\lim_i x_n^{(i)} = x_n$ for each n, put $x=\{x_n\}$ we shall now prove that $\lim_i x^{(i)}=x$ and $x \in C_0$, since $\{x^{(i)}\}$ is a Cauchy sequence in C_0 , given $\varepsilon>0$, there exists $n_0 \in N$, such that

$$\bar{d}(x_n^{(i)}, x_n^{(j)}) \leq \varepsilon/5 \qquad \text{for } i, j \geq n_0$$

Taking the limit as $j \to \infty$, we get

$$\bar{d}(x_n^{(i)}, x_n) \leq \varepsilon/5$$

Therefore $\lim_i x^{(i)} = x$. It remains to show that $x \in C_0$, since $x^{(i)} \in C_0$, there exists $x_0^i \in L(R^k)/\sim$ and $n_1 \in N$, such that

$$\bar{d}(x_n^{(i)}, x_0^{(i)}) < \varepsilon/5 \qquad \text{for } n \geq n_1$$

Hence $\bar{d}(x_0^{(i)}, x_0^{(j)}) \leq \bar{d}(x_n^{(i)}, x_n^{(j)}) + d(x_n^{(i)}, x_0^{(i)}) + d(x_n^{(j)}, x_0^{(j)})$

$$\leq \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = 3\varepsilon/5$$

for $i, j \geq \max(n_0, n_1)$

Thus $\{x_0^{(i)}\}$ is a Cauchy sequence in $L(R^k)/\sim$, so by the theorem 2 there exists $x_0 \in L(R^k)/\sim$, such that

$$\bar{d}(x_0^{(i)}, x_0) \leq 3\varepsilon/5 \qquad \text{for } i \geq \max(n_0, n_1)$$

therefore $\bar{d}(x_n, x_0) \leq \bar{d}(x_n^{(i)}, x_n) + \bar{d}(x_n^{(i)}, x_0^{(i)}) + \bar{d}(x_0^{(i)}, x_0)$

$$\leq \varepsilon/5 + \varepsilon/5 + 3\varepsilon/5 = \varepsilon$$

This implies that $x \in C_0$, so (C_0, ρ) is complete.

Theorem 4 . (B_0, ρ) is also a complete metric space.

Proof . It is similary to the proof of theorem 3 .

Theorem 5 . l^p is a complete metric space with the metric h defined by

$$h(x, y) = \left(\sum_{n} d(x_n, y_n)^{\,p} \right)^{1/p}$$

where $x = \{x_n\}_{n=1}^{\infty} \in l^p$ and $y = \{y_n\}_{n=1}^{\infty} \in l^p$

Proof . It is clear that (l^p, h) is a metric space. To show that l^p is complete in this metric, let $\{x^i\}_{i=1}^{\infty}$ be a Cauchy sequence in l^p. Then for each fixed n, $\{x_n^i\}$ is a Cauchy sequence in $L(R^k)/\sim$, since

$$(L(R^k)/\sim, \bar{d})$$

is complete , we have $\lim_i x_n^{(i)} = x_n$ for each n N. Put $x = \{x_n\}_{n=1}^{\infty}$, it can be shown by standard arguments that $\lim_i x^{(i)} = x$ and $x \in l^p$.

References

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