

ON OPERATIONS FOR PROBABILISTIC SETS. Part 2.

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4. Examples.

Let us illustrate our considerations by means of several examples.

Example 4.1. If random vector (X_1, X_2, \dots, X_k) is uniformly distributed and $\perp_k = \perp_{\lambda}(X_1, X_2, \dots, X_k)$, $T_k = T_{\lambda}(X_1, X_2, \dots, X_k)$, then

$$F_{\perp_k}(z) = \int_0^{t_1} \left(\dots \left(\int_0^{t_i} \dots \left(\int_0^{t_{k-1}} z_k dx_{k-1} \right) \dots dx_i \right) \dots \right) dx_1,$$

$$F_{T_k}(z) = z_1 + \sum_{i=1}^{k-1} \int_{z_i}^1 \left(\dots \left(\int_{z_{i+1}}^1 z_{i+1} dx_i \right) \dots \right) dx_1,$$

here

$$t_1 = z, \quad t_i = g^{-1} \left(\frac{g(t_{i-1}) - g(x_{i-1})}{1 + \lambda g(x_{i-1})/g(z)} \right),$$

$$z_1 = z, \quad z_i = f^{-1} \left(\frac{f(z_{i-1}) - f(x_{i-1})}{1 + \lambda f(x_{i-1})/f(z)} \right).$$

Particularly, if $g(x) = x^2$, $f(x) = (1-x)^2$, $\lambda = 0$, then

dual Yager's operations [16] with parameter $p=2$

are obtained, i.e.

$$\perp_k = \min(1, X_1^2 + X_2^2 + \dots + X_k^2)^{\frac{1}{2}},$$

$$T_k = 1 - \min(1, (1-X_1)^2 + (1-X_2)^2 + \dots + (1-X_k)^2)^{\frac{1}{2}},$$

$$n(x) = 1-x$$

and

$$F_{L_K}(z) = c(K) z^K, \quad F_{T_K}(z) = 1 - c(K) (1-z)^K,$$

where

$$c(K) = \begin{cases} \frac{1}{K!} \left(\frac{\pi}{4}\right)^K, & \text{if } K \text{ is even,} \\ \frac{1}{K!!} \left(\frac{\pi}{2}\right)^K, & \text{if } K \text{ is odd.} \end{cases}$$

Example 4.2. The distribution function of L_K, T_K from example 4.1. for $K=2$ can be rewritten as

$$F_{L_2}(z) = \int_0^z g^{-1} \left(\frac{g(z) - g(x)}{1 + \lambda g(x)/g(z)} \right) dx,$$

$$F_{T_2}(z) = z + \int_z^1 f^{-1} \left(\frac{f(z) - f(x)}{1 + \lambda f(x)/f(z)} \right) dx.$$

Now let's consider the following particular cases:

1. If $f(x) = -\ln x$, $g(x) = -\ln(1-x)$, $\lambda = 0$,

then we have n - dual algebraic operations

$$T_2 = T(x_1, x_2) = x_1 x_2,$$

$$\perp_2 = \perp(x_1, x_2) = x_1 + x_2 - x_1 x_2, \quad n(x) = 1-x.$$

Hence

$$F_{T_2}(z) = z(1 - \ln z), \quad F_{\perp_2}(z) = 1 - (1-z)(1 - \ln(1-z)).$$

2. If $g(x) = x$, $f(x) = 1-x$,

then we have n - dual Sugeno's operations [7]:

$$\perp_2 = \perp(x_1, x_2) = \min(1, x_1 + x_2 + \lambda x_1 x_2),$$

$$\top_2 = \top(x_1, x_2) = \max(0, (1-\lambda)(x_1 + x_2 - 1) + \lambda x_1 x_2),$$

$$n(x) = 1-x$$

and hence

$$F_{\perp_2}(z) = \frac{1}{\lambda} \left(z + \frac{1}{\lambda} \right) \ln(1+\lambda z) - \frac{1}{\lambda} z,$$

$$F_{\top_2}(z) = 1 - \frac{1}{\lambda} \left(1 - z + \frac{1}{\lambda} \right) \ln(1+\lambda(1-z)) - \frac{1}{\lambda} (1-z).$$

5. On probabilistic sets and averaging operators.

In this section we are considering some properties of averaging operators in the sense of the work [8]. Let two-place real function $N: \mathbb{J} \times \mathbb{J} \rightarrow \mathbb{J}$ satisfies the following properties: c), e) continuous,

f), $N(0,0)=0$, $N(\beta,\beta)=\beta$, $N \in \{\min, \max\}$,

g) $N(r_1, r_2) \in [\min(r_1, r_2), \max(r_1, r_2)]$,

and M is averaging operator (not possibly without fail commutative).

Theorem 5.1. The two-place real functions

$$M_g: [0,1] \times [0,1] \rightarrow [0,1],$$

$$M_g(x_1, x_2) = g^{-1}(N(g(x_1), g(x_2))),$$

$$M_f: [0,1] \times [0,1] \rightarrow [0,1],$$

$$M_f(x_1, x_2) = f^{-1}(N(f(x_1), f(x_2)))$$

are averaging operators.

Corollary 5.1. If n is strong negation function then

$$M_n(x_1, x_2) =$$

$$= n^{-1} \circ g^{-1} (N(g \circ n(x_1), g \circ n(x_2)))$$

is averaging operator.

Now let's introduce three definitions.

Definition 5.1. If

$$M_1(x_1, x_2) = n(M_2(n(x_1), n(x_2)))$$

and conversely

$$M_2(x_1, x_2) = n(M_1(n(x_1), n(x_2))),$$

then the averaging operators M_1, M_2 we shall name dual ones regardly to strong negation n .

Definition 5.2. If

$$M(x_1, x_2) = n(M(n(x_1), n(x_2))),$$

then averaging operator M we shall name self-dual regardly to strong negation n .

Example 5.1. M_h, M_g are averaging operators, dual in regard to strong negation n .

Example 5.2. Let $\beta_1 + \beta_2 = 1, \beta_i > 0$, then

$$M_\lambda(x_1, x_2) = g^{-1} \left(\frac{(g(1) + \lambda g(x_1))^{\beta_1} (g(1) + \lambda g(x_2))^{\beta_2} - g(1)}{\lambda} \right)$$

is averaging operator and M_λ is self-dual regardly negation

$$n_\lambda(x) = g^{-1} \left(\frac{g(1) - g(x)}{1 + \lambda g(x)/g(1)} \right).$$

Particularly

$$M_0(x_1, x_2) = \lim_{\lambda \rightarrow 0} M_\lambda(x_1, x_2) = g^{-1}(\beta_1 g(x_1) + \beta_2 g(x_2)),$$

$$M_\infty(x_1, x_2) = \lim_{\lambda \rightarrow \infty} M_\lambda(x_1, x_2) = g^{-1}(g^{\beta_1}(x_1) g^{\beta_2}(x_2)),$$

$$\begin{aligned} M_{-1}(x_1, x_2) &= \lim_{\lambda \rightarrow -1} M_\lambda(x_1, x_2) = \\ &= g^{-1}(g(1) - (g(1) - g(x_1))^{\beta_1} (g(1) - g(x_2))^{\beta_2}). \end{aligned}$$

Hence, new example of averaging operators dual in regard to negation

$$n(x) = g^{-1}(g(1) - g(x)) \quad \text{are } M_{-1}(x_1, x_2), M_\infty(x_1, x_2).$$

Definition 5.3. The averaging operator $M(x_1, x_2)$ satisfies the property of the union (resp: intresection) [8] i.e.

$$M(0, 1) = M(1, 0) = 1,$$

$$(\text{ resp: } M(0, 1) = M(1, 0) = 0),$$

we shall name \mathcal{U} - averaging operator, M_U (resp: \mathcal{I} - averaging operator, M_I), otherwise we shall his name pure averaging operator M_P .

Note, that in [8] notions pure intersection and pure union operators were introduced.

Example 5.3.

$$M_P(x_1, x_2) = M_0(x_1, x_2),$$

$$M_U(x_1, x_2) = M_{-1}(x_1, x_2),$$

$$M_I(x_1, x_2) = M_\infty(x_1, x_2).$$

By definition 5.2. any self-dual averaging operator is not U - or I -averaging operator and hence it is a pure averaging operator. Conversely, if

$$M_p(x_1, x_2) = n(M(n(x_1), n(x_2))),$$

then M is M_p or it is other pure averaging operator, but not U - or I -averaging operators.

Further, if

$$M_I(x_1, x_2) = n(M(n(x_1), n(x_2))),$$

then M is U -averaging operator, but not I -operator.

At last, if

$$M_U(x_1, x_2) = n(M(n(x_1), n(x_2))),$$

then M is I -averaging operator, but not U -operator.

Now, for probabilistic sets A_1, A_2, \dots, A_K

averaging operator will be given by equations:

$$M_g(A_1, A_2, \dots, A_K)(u, \omega) =$$

$$= g^{-1}(Q(g(A_1(u, \omega)), \dots, g(A_K(u, \omega)))),$$

$$M_f(A_1, A_2, \dots, A_K)(u, \omega) =$$

$$= f^{-1}(Q(f(A_1(u, \omega)), \dots, f(A_k(u, \omega))))$$

for all $u \in U, \omega \in \Omega$, where Q is k -place real function satisfies c), e) and

$$Q(0, 0, \dots, 0) = 0, Q(\beta, \beta, \dots, \beta) = \beta, Q \in \{\min, \max\},$$

$$Q(r_1, r_2, \dots, r_k) \in [\min_{i=1, k} r_i, \max_{i=1, k} r_i].$$

Particularly we obtain

$$M_\lambda(A_1, A_2, \dots, A_k)(u, \omega) =$$

$$= g^{-1}\left(\frac{1}{\lambda} \left(\prod_{i=1}^k (g(1) + \lambda g(A_i(u, \omega)))^{\beta_i} - g(1) \right)\right),$$

$$\beta_1 + \beta_2 + \dots + \beta_k = 1, \beta_i > 0,$$

for all $u \in U, \omega \in \Omega$.

Statements, similar to Theorem 3.1. or Corollary 3.1., can be made for averaging operator. Let

$$M_I = M_I(x_1, x_2, \dots, x_k), M_U = M_U(x_1, x_2, \dots, x_k).$$

Theorem 5.2. If M_I, M_U are dual in regard to strong negation $n(\infty) = 1 - \infty$, then

$$F_{M_I}(z) + F_{M'_U}(1-z) = 1,$$

where $M'_U = M_U(1-x_1, 1-x_2, \dots, 1-x_k)$.

Corollary 5.2. If M_I, M_U are dual in regard to negation $\eta(x) = 1-x$ and random vector (X_1, X_2, \dots, X_k) is uniformly distributed, then

$$F_{M_I}(z) + F_{M_U}(1-z) = 1.$$

Example 5.4. Let

$$M_I = x_1^{\beta_1} x_2^{\beta_2}, \beta_1 \neq \beta_2, \beta_1 + \beta_2 = 1, \beta_i > 0,$$

$$M_U = 1 - (1-x_1)^{\beta_1} (1-x_2)^{\beta_2}$$

If random vector (X_1, X_2) is uniformly distributed, then

$$\begin{aligned} F_{M_I}(z) &= \iint_{D(z)} \psi_{x_1 x_2}(x_1, x_2) dx_1 dx_2 = \\ &= \int_0^{z^{\beta_1^{-1}}} dx_1 + \int_{z^{\beta_1^{-1}}}^1 (z x_1^{-\beta_1})^{\beta_2^{-1}} dx_1 = \frac{\beta_1}{\beta_2 - \beta_1} (z^{\beta_2^{-1}} - z^{\beta_1^{-1}}), \end{aligned}$$

where

$$D(z) = \{(x_1, x_2) : x_1 x_2 < z\}.$$

Hence, according to Corollary 5.2.

$$\begin{aligned} F_{M_U}(z) &= 1 - F_{M_I}(1-z) = \\ &= 1 - \frac{\beta_2}{\beta_2 - \beta_1} ((1-z)^{\beta_2^{-1}} - (1-z)^{\beta_1^{-1}}). \end{aligned}$$

Similarly, as section 3, the distribution function representation of averaging operators for independent probabilistic sets is obtained, but missing here.

R e f e r e n c e s

- [1] F.B.Abutaliev, R.Z.Salakhutdinov, On fuzzy set operations based on triangular norms, *Isvestiya AN UzSSR seriya techn.* 4 (1988) 9-13 (in Russian)..
- [2] F.B.Abutaliev, R.Z.Salakhutdinov, On fuzzy set operations, *BUSEFAL* (to appear).
- [3] E.Czogaia, On distribution function description of probabilistic sets and its application in decision making, *Fuzzy sets and Systems* 10 (1983) 21-29.
- [4] E.Czogaia, S.Gottvald, W.Pedrycz, Logical connectives of probabilistic sets, *Fuzzy sets and Systems* 10 (1983) 299-308.
- [5] E.Czogaia, A generalized concept of a fuzzy probabilistic controller, *Fuzzy sets Systems* 11 (1983) 287-297.
- [6] E.Czogaia, H.-J. Zimmermann, The aggregation operations for decision making in probabilistic fuzzy environment, *Fuzzy sets and Systems* 13 (1984) 223-239.
- [7] D.Dubois, H.Prade, A class of fuzzy measures based on triangular norms, *Int. J. General Systems* 8 (1982) 43-61.
- [8] D.Dubois, H.Prade, New results about properties and semantics of fuzzy set-theoretic operators, In: *Fuzzy sets. Theory and Applications to Policy Analysis and Information*

Systems, edited by P.P.Wang, S.K.Shang, Plenum Press, N.Y.
(1980) 59-75.

- [9] K.Hirota, Concepts of probabilistic sets, Fuzzy sets and Systems 5 (1981) 31-46.
- [10] H.Kwakernaak, Fuzzy random variables - 1, Definition and theorem, Information Sciences 15 (1978) 1-29.
- [11] H.Kwakernaak, Fuzzy random variables- 2, Algorithms and examples for the discrete case, Information Sciences 17 (1979) 253-278.
- [12] S.Nahmias, Fuzzy variables, Fuzzy sets and Systems 1 (1978) 97-110.
- [13] B.Schweizer, A.Sklar, Associative functions and statistical triangle inequalities, Publicationes Mathematicae Debrecen 8 (1961) 169-186.
- [14] M.Sugeno, Fuzzy measures and integrals - survey, In: Fuzzy Automata and Decision Process, edited by M.M.Gupta, B.N.Saridis B.R.Gaines, North-Holland, Amsterdam (1977) 89-102.
- [15] E.Trillas, C.Alsina, L.Valverde, Do we need max, min and 1-j in fuzzy sets theory?, In: Fuzzy Set and Possibility Theory. Recent Developments, edited by R.R.Yager, Pergamon Press. N.Y. (1980) 275-300.
- [16] R.R.Yager, On a general class of fuzzy connectives, Fuzzy sets and Systems 4 (1980) 235-242.
- [17] L.A.Zadeh,Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications 12 (1968) 94-102.