FUZZY LINEAR SYSTEMS*

Ketty PEEVA

Sofia 1000, P.O.Box 384, Inst. of Appl. Math., Technical University

Let $\mathbb{L}=(L,\vee,\wedge,0,1)$ be a bounded chain, $I,J\neq\emptyset$ be finite sets and $a: I\times J \longrightarrow L$ be a map. $A=(a_{ij})\in L^{I\times J}$ is a matrix over L if $a_{ij}=a(i,j)$, $\forall i\in I$, $\forall j\in J$. The product $C=A.B=(c_{ij})\in L^{I\times J}$ is defined with $c_{ij}=\vee_{k\in K}(a_{ik}\wedge b_{kj})$, $\forall i\in I, \forall j\in J$.

In what follows $\{1\}$ stand for the singleton set, $|I|=m\in\mathbb{N}$, $|J|=n\in\mathbb{N}$. The matrices A, X and B denote coefficients, unknowns and constants respectively. We shall write $A.X=B$, $A.X\gneqq B$, $A.X> B$, $A.X\lneqq B$, $A.X< B$ for (1),(2),(3),(4),(5) respectively:

$\vee_{j\in J}\ (a_{ij}\wedge x_j)=b_i,\ i\in I$ (1)

$\vee_{j\in J}\ (a_{ij}\wedge x_j)\gneqq b_i, i\in I$ (2) $\qquad \vee_{j\in J}(a_{ij}\wedge x_j)> b_i,\ i\in I$ (3)

$\vee_{j\in J}\ (a_{ij}\wedge x_j)\lneqq b_i, i\in I$ (4) $\qquad \vee_{j\in J}(a_{ij}\wedge x_j)< b_i,\ i\in I$ (5)

where $A=(a_{ij})\in L^{I\times J}$, $X=(x_j)\in L^{J\times\{1\}}$, $B=(b_i)\in L^{I\times\{1\}}$. If the statement is valid for any of (1) $\div$ (5) we write $A.X\perp B$ or $(\perp)$.

$X^0=(x^0_j)\in L^{J\times\{1\}}$ is a point solution of $(\perp)$ if $A.X^0\perp B$ holds; P^0 is the set of all point solutions of $(\perp)$. If $P^0\neq\emptyset$, then $(\perp)$ is consistent, otherwise it is inconsistent. $\underline{X}^0\in P^0$ is a lower point solution of $A.X\perp B$ if for any $X^0\in P^0$ the relation $X^0\lneqq\underline{X}^0$ implies $X^0=\underline{X}^0$. The upper point solution is defined dually. $X_j\subseteq L$ is called feasible, if the choise of any $x_j\in X_j$ does not result in a contradiction in $(\perp)$. An n-tuple $(X_1,\dots,X_n)$ of feasible intervals is an interval solution of $(\perp)$ if any $X^0=(x^0_j)$ with $x^0_j\in X_j$ belongs to P^0.

We assign to $A.X\perp B$ a new system $A^*.X\perp B$, where $A^*=(a^*_{ij})$ is computed from A with respect to B: $a^*_{ij}=G$ if $a_{ij}> b_i$; $a^*_{ij}=E$ if $a_{ij}=b_i$; $a^*_{ij}=S$ if $a_{ij}< b_i$.

LEMMA. The systems $A.X\perp B$ and $A^*.X\perp B$ are equivalent.

Let the system $A.X\perp B$ be given and the j^{th} column in A^* be fixed; k stand for the greatest number of the row with G-type coefficient and r - for the smallest number of the row with E-type coefficient in the j^{th} column of A^*.

THEOREM 1. Let the system $A.X = B$ be given.

i) If the j^{th} column of A^* contains G-type coefficient a^*_{kj}, then:

* The complete text will appear in Fuzzy Sets and Systems

a) $X_j=[0,b_k]$ is a feasible interval for the j^{th} component;
b) $x_j=b_k$ implies $a_{ij}\wedge x_j=b_i$ for $i=k$, for each $i<k$ with $a_{ij}>b_i=b_k$ and for each $i>k$ with $a_{ij}=b_i$;
ii) if the j^{th} column of A^* does not contain any G-type coefficient, but it contains E-type coefficient $a^*_{rj}=b_r$, then:
a) $X_j=L$ is a feasible interval for the j-th component;
b) $x_j\in[b_r,1]$ implies $a_{ij}\wedge x_j=b_i$ for each $i\geqslant r$ with $a_{ij}=b_i$;
iii) if the j^{th} column of A^* does not contain neither G, nor E-type coefficient, then $X_j=L$ is feasible and $a_{ij}\wedge x_j<b_i$ holds for any x_j.

THEOREM 2. Let the system $A.X\geqslant B$ be given.
i) If the j^{th} column of A^* contains G-type coefficient a^*_{kj}, then:
a) $X_j=[b_k,1]$ is a feasible interval for the j^{th} component;
b) $x_j\in[b_k,1]\Rightarrow a_{ij}\wedge x_j\geqslant b_i$ for $i=k$, for each $i<k$ with $a_{ij}>b_i=b_k$;
c) for $i>k$: if $a_{ij}=b_i$, then $x_j\in[b_r,1]$ means $a_{ij}\wedge x_j=b_i$;
ii) if the j^{th} column of A^* does not contain any G-type coefficient, but it contains E-type coefficient $a^*_{rj}=b_r$, then:
a) $X_j=[b_r,1]$ is a feasible interval for the j^{th} component;
b) $x_j\in[b_r,1]$ implies $a_{ij}\wedge x_j=b_i$ for each $i\geqslant r$ with $a_{ij}=b_i$ and there does not exist $x_j\in L$ such that $a_{ij}\wedge x_j>b_i$;
iii) if the j^{th} column of A^* contains only S-type coefficients, then $a_{ij}\wedge x_j<b_i$ for each $x_j\in L$.

THEOREM 3. Let the system $A.X\leqslant B$ be given.
i) If the j^{th} column of A^* contains G-type coefficient a^*_{kj}, then for any $x_j\in[0,b_k]$ and: for $i=k$ the inequality $a_{ij}\wedge x_j\leqslant b_i$ holds; for $i<k$ and $a_{ij}>b_i$ the inequality $a_{ij}\wedge x_j\leqslant b_i$ holds; for any E-type coefficient $a^*_{ij}=b_i$, the inequality $a_{ij}\wedge x_j\leqslant b_i$ holds for any $i\in I$;
ii) if the j^{th} column in A^* does not contain any G-type coefficient, but it contains E-type coefficient, then $X_j=L$ is a feasible interval and $x_j\in L$ means $a_{ij}\wedge x_j\leqslant b_i$ for each $i=1,\dots,m$.
iii) if the j^{th} column in A^* contains only S-type coefficients, then $a_{ij}\wedge x_j<b_i$ for each $i=1,\dots,m$ and any $x_j\in X_j=L$.

For $A.X>B$ and $A.X<B$ a slight modification of these results is valid.

COROLLARY: The following problems are algorithmically decidable for ($\perp$): is the system consistent or not; if it is consistent - computing the greatest point solution, the lower point solutions, the maximal interval solutions; marking the numbers of the contradictory equations, if the system is inconsistent.