SOFT FUZZY σ-ALGEBRAS

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We generalize some notions of the classical measure theory to the fuzzy case preserving as much classical properties as possible. We use the Zadeh's fuzzy connectives. Let $X \neq \emptyset$ be a universum. Then

S is a σ-algebra	σ is fuzzy σ-algebra [1]
$\emptyset, X \in S$	$0_X, 1_X \in \sigma$
$A \in S \Longrightarrow A^c \in S$	$\mu \in \sigma \Longrightarrow \mu' \in \sigma$
$\{A_n\} \subset S \Longrightarrow \bigcup A_n \in S$	$\{\mu_n\} \subset \sigma \Longrightarrow \bigvee \mu_n \in \sigma$

The next notions (in fuzzy set theory) are due to Piasecki.

$A \cap A^c = \emptyset$	empty set	$\mu \wedge \mu'$	W-empty set
$A \cup A^c = X$	universum	$\mu \vee \mu'$	W-universum

$W_0(\sigma)$ is the system of all W-empty sets of σ

$W_1(\sigma)$ is the system of all W-universums of σ

$A \cap B = \emptyset$ disjoint sets $\quad \mu \wedge \eta \in W_0(\sigma)$ W-disjointness

The original Piasecki's W-disjointness follows from A is disjoint with B iff $A \subset B^c$ [3] , i.e. μ is W-disjoint with η iff $\mu \leq \eta'$ (i.e. the Lukasiewicz's conjunction of μ and η is 0_X). However, both principles lead to the same results. It is natural to demand no W-empty set be a W-universum, simultaneously, i.e. $W_0(\sigma) \cap W_1(\sigma) = \emptyset$. This is equivalent to the condition $(1/2)_X \notin \sigma$. If $(1/2)_X$ is not contained in a fuzzy σ-algebra σ, then is called a soft fuzzy σ-algebra.

<u>Example 1</u>. Let σ be any fuzzy σ-algebra and let $\mu \in W_1(\sigma)$, $\mu \neq (1/2)_X$. Then $\sigma_\mu = \{\eta \in \sigma, \eta \vee \eta' \geq \mu\}$ is a soft fuzzy σ-algebra and μ is a lower bound of the system $W_1(\sigma_\mu)$.

It is easy to see that the smallest soft fuzzy sub-σ-algebra of σ is $\{0_X, 1_X\}$ and that the greatest one needn't exist. We ask if any soft fuzzy σ-algebra σ is of a similar form as σ_μ , i.e. there is (in some sense) a lower bound of $W_1(\sigma)$ contained in $W_1(\sigma)$. This is not true (see Example 2.), in general, nor in the form presented in Example 1., neither in a weaker form: there is $\mu \in W_1(\sigma)$ such that for any $\eta \in W_1(\sigma)$ there is $\tau \in \sigma$, $\{\eta < \mu\} \subset \{\tau = 1/2\}$.

Example 2. Let $X = [0,1]$, $\sigma = \{\mu, \mu(x) \in \{0, \frac{1}{4}, \frac{3}{4}, 1\}$, $\mu(x) \in \{\frac{1}{4}, \frac{3}{4}\}$ at most in countably many x's $\}$. Then σ is a soft fuzzy σ-algebra with no lower bound of $W_1(\sigma)$ of above mentioned types.

Some results for σ being a soft fuzzy σ-algebra [2],[4]: $K(\sigma) = \{A \subset X, \exists \mu \in \sigma : \{\mu > 1/2\} \subseteq A \subseteq \{\mu \geq 1/2\}\}$ is a σ-algebra of crisp subsets of X. Let $\mu \in W_1(\sigma)$. Then $\sigma_\mu^* = \{\eta \in \sigma, \eta \vee \eta' = \mu\} \cup \{0_X, 1_X\}$ is a soft fuzzy σ-algebra. We have $\sigma = \bigcup_{W_1(\sigma)} \sigma_\mu^*$, $K(M) = \bigcup_{W_1(\sigma)} K(\sigma_\mu^*)$, $\sigma_\mu = \bigcup_{\eta \geq \mu} \sigma_\eta^*$.

Let f be a random variable on $(X, K(\sigma))$. Then there is $\mu \in W_1(\sigma)$ such that f is $K(\sigma_\mu^*)$-measurable. The same is true for any denumerable system $\{f_n\}$ of random variables on $(X, K(\sigma))$. This fact is of great importance in the theory of fuzzy observables [5]. For more details see [2].

probability measure P on (X,S)	fuzzy P-measure p on (X,σ)
$P(X) = 1$, $X = A \cup A^c$	$p(\mu \vee \mu') = 1$
$\{A_n\} \subset S$ pairwise disjoint	$\{\mu_n\} \subset \sigma$ pairwise W-disjoint
$P(\bigcup A_n) = \sum P(A_n)$	$p(\bigvee \mu_n) = \sum p(\mu_n)$

It is easy to see that σ must be a soft fuzzy σ-algebra.

Problem (Dvurečenskij): Let σ be a soft fuzzy σ-algebra. Does there exist a fuzzy P-measure p on (X,σ)?

If $\bigcup_{\mu \in \sigma} \{\mu = 1/2\} \neq X$, then the answer is positive. In general, this problem remains open. p on (X,σ) is equivalent to P on $(X, K(\sigma))$, $p(\mu) = P(\mu > 1/2)$ and $P(A) = p(\mu)$ if $\{\mu > 1/2\} \subseteq A \subseteq \{\mu \geq 1/2\}$. More, for any $\mu \in \sigma$, $P(\mu = 1/2) = 0$.

Hypothesis. Let σ be a soft fuzzy σ-algebra and let p be a fuzzy P-measure on σ. Then there is $\mu \in W_1(\sigma)$ such that $K(\sigma_\mu^*) = K(\sigma)$, P-a.e., i.e. for any $A \in K(\sigma)$ there is $B \in K(\sigma_\mu^*)$ such that $P(A \cup B) = P(A \cap B)$.

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