CONTINUOUS LATTICES OF FUZZY SUBALGEBRAS AND CONGRUENCE RELATIONS

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By now the theory of algebraic lattices and the associated notion of algebraic closure systems and operators, and the lattice of subalgebras of an algebra has been well developed and has become classical work. Continuous lattices have risen in the study of many different context, viz. theory of computation, logic, topology, spectral theory of rings. They also represent a generalisation of algebraic lattices. The following illustrates the point.

CRISP CASE: 1.Algebraic lattices; 2.Lattices of subalgebras; 3.Algebraic closure systems; 4. Algebraic closure operators; 5. Lattices of 2^x closed under arbitrary infirmums and supremums of directed sets.

FUZZY CASE: 1. Algebraic lattices in I^x; 2.Lattices of fuzzy subalgebras; 3. Algebraic closure systems in I^x; 4. Algebraic clossure operations in I^x; 5. Lattices of I^x closed under arbitrary infimums and supremums of directed sets.

Numbers 1 to 5 in the crisp case are equivalent. Numbers 1 to 4 in the fuzzy case have been considered by us in earlier papers [2], [3].

The subject of discussion of this paper is to prove that numbers 1 to 4 imply 5 in the fuzzy case; thus proving that the lattice of fuzzy subalgebras is a continuous lattice. This raises the open problem of characterising any given continuous lattice as isomorphic to a lattice of fuzzy subalgebras.

What sort of order relation represent the property of subsets $\mathcal{A}$ in I^x closed under arbitrary infirmums and supremums of directed sets? The usual less than or equal $\leq$ or the usual inclusion is not strong enough. An order relation suitable to characterise the above property is called way below relation denoted by <<. In a complete lattice L (say I^X), for x, y $\in$ L, x is way below y, x << y if and only if for every directed subset D of L the relation y $\leq$ sup D implies there exists a d $\in$ D such that x $\leq$ d. An element x $\in$ L with x << x is said to be compact or isolated from below. Observe that in the unit interval 0 is the only compact element. In the fuzzy case for $\mu, \nu \in I^X$, $\mu << \nu$ if and only if $\mu(x) < \nu(x)$ for all x $\in$ X, and $\mu(x) \neq 0$ for only a finite number of x $\in$ X. Now a complete lattice L is called a continous lattice if every x $\in$ L can be written as $x = \sup \{ u \in L: u << x\}$. With this definition, it is proved in [1] that L is continuous if and only if L is isomorphic to a subset of I^X which closed under arbitrary infimums and directed supremums. In the case when X is an algebra, that subset of I^X closed under arbitrary infs and directed sups is the set of all fuzzy subalgebras of X or the set of fuzzy congruence relations on X. Thus the lattice of fuzzy subalgebras of X is a continuous lattice. More technical details can be found in [2], [4].

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1. G Gierz, K.H. Hofmann et al, A Compendium of Continuous Lattices, Springer–Verlag, Berlin, 1980.

2. V Murali, on lattice of fuzzy subalgebras and closure systems in I^X, to appear in fuzzy set and system.

3. V Murali, on fuzzy congruence relations, to appear in fuzzy set and system.

4. V Murali, Continuous lattices of fuzzy subalgebras, preprint, submitted.

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