FUZZY SETS HELP UNDERSTAND CRISP DECISION THEORY

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1. **Motivation.** Fuzzy constructions too often have no influence on crisp prototypes. The desirable interaction should be mutual – say, **CRISP** $\xrightarrow{\text{generalization}}$ **FUZZY**, $\xleftarrow[\text{explanation, new concepts}]{}$. Below, several examples of this kind within the frame of decision theory with binary relations are demonstrated (**R** is [crisp or fuzzy] antireflexive binary preference relation (**BR**) on a finite set **X** of alternatives).

2. **Rationality concepts.** Crisp choice rules with **BR**s are based on "rationality concepts" (**RC**) $p(R):2^X \longrightarrow \{0,1\}$, yielding "multifold choice" $\mathcal{D}_c(p,R)=p(R)^{-1}(1)$ [1]. Algebraic formulas of three basic **RC**'s [3–6] are as follows(**Z⊆X is rational iff <Formula>**) :

Name	**R-non-domination**	**Independence**	**Domination**
Denotation	$\Delta_1(R)(Z)$	$\Delta_2(R)(Z)$	$\Delta_3(R)(Z)$
Formula	$R\square\bar{Z}\subseteq\bar{Z}$	$R\square Z\subseteq\bar{Z}$	$\bar{Z}\subseteq R\square Z$

(in this table, □ is **composition law** – boolean product).
Set $\Pi_\Delta(\square)=\{$**non-decreasing boolean polynoms** $p(\Delta_1,\Delta_2,\Delta_3)\}$.
This family of **RC**'s is free distributive lattice with three generators (D_{18}), containing **18** decision procedures. **Dual composition law** $|\bar{\square}$, $R|\bar{\square}Z=\overline{R\square Z}$ generates alternative family $\Pi_\Delta(|\bar{\square})$. Many of well-known choice rules are included in $\Pi_\Delta(\square)$: **non-domination ND** – $\Delta_1\wedge\Delta_2$; **von Neumann – Morgenstern solution NMS** – $\Delta_2\wedge\Delta_3$; **game-theoretical kernel GTK** $\Delta_1\wedge\Delta_2\wedge\Delta_3$, **GOCHA** [7] – Δ_1. $\Pi_\Delta(|\bar{\square})$ contains **RC** Δ_3, underlying **GETCHA** rule [7].

3. **Fuzzy decision procedures. Optimal solutions.** Fuzzy generalization of **RC** is **fuzzy decision procedure FDP** [3,5,6] $p:\mathcal{P}(X^2)\longrightarrow\mathcal{P}^{(2)}(X)$, $p(R)$ being **fuzzy level 2 subset** of X. Fuzzy versions of Δ_1 depend on **fuzzy inclusion inc** and **composition law** $\odot$ ($\mu_{\Delta_1(R)}(a)=\mu_{inc}(R\odot\bar{a},\bar{a})$; $\mu_{\Delta_2(R)}(a)=\mu_{inc}(R\odot a,\bar{a})$ etc.). $\Pi_\Delta(\odot,inc)$ is defined as in crisp case. "Truth set" $\mathcal{D}_c(p,R)$ is changed for **optimal solution** $\mathcal{D}_f(p,R)=\mu^{-1}_{p(R)}(\mu^*_{p(R)})$, $\mu^*_{p(R)}=\bigvee_{a\in\mathcal{P}(X)}\mu_{p(R)}(a)$.

4. **Comparative study of choice rules.** Suppose $inc=I_5$ is Dienes inclusion. Fuzzy **ND** rule $p=(\Delta_1\wedge\Delta_2)(\square,I_5)$ has

optimal solution $\mathcal{D}_f(p,R)=[0,\chi_{ND(R_{>0})}]$, containing $\varepsilon\circ\chi_Z$ with any $Z\subseteq ND(R_{>0})$ and arbitrarily small ε as well as $ND(R_{>0})$ itself. Similarly, **16** of **18** choice rules in $\Pi_\Delta(\square,inc)$ ($inc\in\{I_5,\subseteq\}$ [2]) induce **no confident crisp choice**). Only **NMS** and **GTK** yield "meaningful" fuzzy solutions [3,4]; for **NMS** $p=\Delta_2\wedge\Delta_3$, $\mathcal{D}_f(p,R)=\bigcup_{K\in\mathcal{K}^*}[\mu^*\chi_K,\chi_K\vee\bar{\mu}^*\chi_{\bar{K}}]$; ($\mu^*=\mu^*(p,R)>1/2$; $\mathcal{K}^*=\{K\in NMS(R_{>1/2})\mid\mu_{p(R)}(\chi_K)=\mu^*\}$). So, optimal solution is collection of **fuzzy dichotomies** $[\mu^*,1]/K+[0,\bar{\mu}^*]/K$, providing stable **guaranteed resolvability** $\mu^*-\bar{\mu}^*>0$, in contrast with **ND**-case.

5. **Adjusting crisp choice**. When applied to **crisp** relations, **FDPs** often change **dichotomy** *chosen/rejected* to **triangulation** *chosen/uncertain/rejected*. Thus, with **GETCHA** rationality concept [7] $p=\Delta_3(\mid\bar{\square},I_5)$, triangulation is $\tau_Z=\{Z\setminus R\mid\bar{\square}Z,\ Z\cap R\mid\bar{\square}Z,\ Z\}$ ($Z\in\mathcal{D}_c(p,R)$); so, **FDPs** "automatically" turn boolean logic into the three-valued one.

6. **Producing new choice rules**. "Exact fuzzy **NMS**" [4] $p=(\Delta_2\wedge\Delta_3)(\square,\subseteq)$; $\mathcal{D}_f(p,R)=\{a\mid R\square a=\bar{a}\}$ introduces new **crisp ranking concept** – **"dipole decomposition"**, that is, a partition $\delta=\{Z_k^+,Z_0,Z_k^-\}_1^m$ of the support X, satisfying the system of equations: $R\square Z_k^+=Z_k^-$, $R^{-1}\square Z_k^+\subseteq\bigcup_{j=1}^{k}Z_j^-$ and being the generalization of von Neumann – Morgenstern solution.

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